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The Raney numbers and $(s, s+1)$-core partitions

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ABSTRACT

The Raney numbers $R_{p,r}(k)$ are a two-parameter generalization of the Catalan numbers. In this paper, we give a combinatorial proof for a recurrence relation of the Raney numbers in terms of coral diagrams. Using this recurrence relation, we confirm a conjecture posed by Amdeberhan concerning the enumeration of $(s, s+1)$-core partitions λ with parts that are multiples of p. As a corollary, we give a new combinatorial interpretation for the Raney numbers $R_{p+1,r+1}(k)$ with $0 \leq r < p$ in terms of $(kp+r, kp+r+1)$-core partitions λ with parts that are multiples of p.

1. Introduction

In this paper, we build a connection between the Raney numbers and $(s, s+1)$-core partitions with parts that are multiples of p. We show that the number of $(kp+r, kp+r+1)$-core partitions with parts that are multiples of p equals the Raney number $R_{p+1,r+1}(k)$, confirming a conjecture posed by Amdeberhan [1].

The Raney numbers $R_{p,r}(k)$ were introduced by Raney in his investigation of functional composition patterns [14] and these numbers have also been used in probability theory [11,12]. The Raney numbers $R_{p,r}(k)$ are defined as follows:

$$R_{p,r}(k) = \frac{r}{kp+r}\binom{kp+r}{k}. \tag{1.1}$$

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The Raney numbers are a two-parameter generalization of the Catalan numbers. To be more specific, if $r = 1$, the Raney numbers specialize to the Fuss–Catalan numbers $C_p(k)$ [8,9], where $C_p(k)$ are the numbers of p-ary trees with k internal vertices and

$$C_p(k) = R_{p,1}(k) = \frac{1}{kp+1}\binom{kp+1}{k}.$$

If we further set $p = 2$, we obtain the classical Catalan numbers C_k, that is,

$$R_{2,1}(k) = C_k = \frac{1}{k+1}\binom{2k}{k}.$$

Let $\mathcal{C}_p(x)$ and $\mathcal{R}_{p,r}(x)$ denote the generating functions of the Fuss–Catalan numbers $C_p(k)$ and the Raney numbers $R_{p,r}(k)$, respectively, namely,

$$\mathcal{C}_p(x) = \sum_{k\geq 0} C_p(k)x^k = \sum_{k\geq 0} \frac{1}{kp+1}\binom{kp+1}{k}x^k,$$

$$\mathcal{R}_{p,r}(x) = \sum_{k\geq 0} R_{p,r}(k)x^k = \sum_{k\geq 0} \frac{r}{kp+r}\binom{kp+r}{k}x^k.$$

It is easily seen that $\mathcal{C}_p(x) = \mathcal{R}_{p,1}(x)$. The following theorem gives more relations of the generating functions $\mathcal{C}_p(x)$ and $\mathcal{R}_{p,r}(x)$.

Theorem 1.1 ([8,9]). *Let p be a positive integer and let r, k be nonnegative integers. Then we have*

$$\mathcal{C}_p(x) = 1 + x\mathcal{C}_p(x)^p, \tag{1.2}$$

$$\mathcal{R}_{p,r}(x) = \mathcal{C}_p(x)^r. \tag{1.3}$$

Notice that $\mathcal{C}_p(x) = \mathcal{R}_{p,1}(x)$. The following theorem is followed directly by equating the coefficients of x^k in (1.2) and (1.3).

Theorem 1.2. *Let p be a positive integer and let r, k be nonnegative integers. Then the number $R_{p,r}(k)$ satisfies the recurrence relations*

$$R_{p,1}(k) = \sum_{i=0}^{k-1} R_{p,1}(i)R_{p,p-1}(k-1-i), \tag{1.4}$$

$$R_{p,r}(k) = \sum_{i=0}^{k} R_{p,1}(i)R_{p,r-1}(k-i), \quad \text{for } r > 1, \tag{1.5}$$

with the initial values $R_{p,r}(0) = 1$ if $r \geq 0$ and $R_{p,0}(k) = 0$ if $k > 0$.

Notice that $C_k = R_{2,1}(k)$. Substituting $p = 2$ into (1.4), we obtain the recurrence relation for the Catalan numbers $C_k = \sum_{i=0}^{k-1} C_iC_{k-1-i}$.

Let us give an overview of notation and terminology on partitions. A *partition* λ of a positive integer n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that $\lambda_1+\lambda_2+\cdots+\lambda_m = n$. We write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \vdash n$ and we say that n is the *size* of λ and m is the *length* of λ. The *Young diagram* of λ is defined to be an up- and left-justified array of n boxes with λ_i boxes in the ith row. Each box B in λ determines a *hook* consisting of the box B itself and boxes directly to the right and directly below B. The *hook length* of B, denoted $h(B)$, is the number of boxes in the hook of B.

For a partition λ, the *β-set* of λ, denoted $\beta(\lambda)$, is defined to be the set of hook lengths of the boxes in the first column of λ. For example, Fig. 1 illustrates the Young diagram and the hook lengths of a partition $\lambda = (5, 3, 2, 2, 1)$. The β-set of λ is $\beta(\lambda) = \{9, 6, 4, 3, 1\}$. Notice that a partition λ is uniquely determined by its β-set. Given a decreasing sequence of positive integers $(h_1, h_2, \ldots, h_m)$, it is easily seen that the unique partition λ with $\beta(\lambda) = \{h_1, h_2, \ldots, h_m\}$ is $\lambda = (h_1 - (m-1), h_2 - (m-2), \ldots, h_{m-1} - 1, h_m)$.