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A Gröbner basis characterization for chordal comparability graphs

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ABSTRACT

In this paper, we study toric ideals associated with multichains of posets. It is shown that the comparability graph of a poset is chordal if and only if there exists a quadratic Gröbner basis of the toric ideal of the poset. Strong perfect elimination orderings of strongly chordal graphs play an important role.

0. Introduction

An $n \times m$ integer matrix $A = (\mathbf{a}_1, \ldots, \mathbf{a}_m)$ is called a *configuration* if there exists $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{a}_j \cdot \mathbf{c} = 1$ for $1 \leq j \leq m$. Let $K[y_1, \ldots, y_m]$ be a polynomial ring in m variables over a field K. Given a configuration A, the binomial ideal

$$I_A = \left\langle \prod_{b_i>0} y_i^{b_i} - \prod_{b_j<0} y_j^{-b_j} \in K[y_1, \ldots, y_m] : \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{Z}^m, A\mathbf{b} = \mathbf{0} \right\rangle$$

is called the *toric ideal* of A. Any toric ideal is generated by homogeneous binomials, and has a Gröbner basis consisting of homogeneous binomials. See [11,20] for basics on toric ideals. Each of the following is one of the most important and fundamental problems on toric ideals:

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(a) Is the toric ideal I_A generated by quadratic binomials?
(b) Does there exist a monomial order such that a Gröbner basis of I_A consists of quadratic binomials?

Note that any Gröbner basis of I_A is a set of generators of I_A. These problems arise in the study of Koszul algebras. The algebra $K[y_1, \ldots, y_m]/I_A$ is said to be *Koszul* if the minimal graded free resolution of K as a $K[y_1, \ldots, y_m]/I_A$-module is linear. It is known that

$$\begin{aligned} I_A \text{ has a quadratic Gröbner basis } &\Longrightarrow K[y_1, \ldots, y_m]/I_A \text{ is Koszul} \\ &\Longrightarrow I_A \text{ is generated by quadratic binomials} \end{aligned}$$

holds in general. However, all of the converse implications are false. See, e.g., [17]. Problems (a) and (b) are studied for configurations arising from various kinds of combinatorial objects. The following is a partial list of them:

(1) Toric ideals arising from order polytopes of finite posets [10];
(2) Toric ideals arising from cut polytopes of finite graphs [6,14];
(3) Toric ideals of the vertex–edge incidence matrix of finite graphs [16,17];
(4) Toric ideals arising from graphical models [5,8];
(5) Toric ideals arising from matroids [2,3,12,13].

In particular, one of the most famous open problems on toric ideals is White's conjecture [22]: He conjectured that the toric ideal arising from any matroid is generated by some quadratic binomials.

In the present paper, we study toric ideals associated with multichains of posets. Let $d \geq 2$ be an integer and let $P = \{x_1, \ldots, x_n\}$ be a poset. We associate a multichain $C : x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_d}$ of length $d-1$ with a (not necessarily (0, 1)) vector $\rho(C) = \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \cdots + \mathbf{e}_{i_d} \in \mathbb{Z}^n$, where $\mathbf{e}_i$ is the ith unit vector in $\mathbb{R}^n$. We often regard C as a multiset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_d}\}$. Let $\mathcal{M}_d(P) = \{C_1, \ldots, C_m\}$ be a set of multichains of P of length $d-1$. Then the toric ideal $I_{\mathcal{M}_d(P)}$ of $\mathcal{M}_d(P)$ is the toric ideal of the configuration $(\rho(C_1), \ldots, \rho(C_m))$. For example, if $d = 3$ and $P = \{x_1, x_2, x_3\}$ is a poset whose maximal chains are $x_1 > x_2$ and $x_2 < x_3$, then the corresponding configuration is

$$\begin{pmatrix} 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix}.$$

For any $d \geq 2$ and chain $P = \{x_1, \ldots, x_n\}$ of length $n-1$, it is known that $I_{\mathcal{M}_d(P)}$ is the toric ideal of the dth *Veronese subring* of a polynomial ring in n variables, and $I_{\mathcal{M}_d(P)}$ has a quadratic Gröbner basis. Thus, in general, $I_{\mathcal{M}_d(P)}$ is a toric ideal of a subconfiguration of the dth Veronese subring. There are several results on toric ideals of subconfigurations of the dth Veronese subring: algebras of Veronese type [20, Theorem 14.2] and algebras of Segre–Veronese type [18,1]. However, the results of the present paper are different from these results. The toric ideal of algebras of Veronese / Segre–Veronese type has a squarefree initial ideal. On the other hand, $I_{\mathcal{M}_d(P)}$ has no squarefree initial ideal except for some trivial cases (Proposition 1.3).

This paper is organized as follows. In Section 1, it is shown that the comparability graph of a poset is chordal if and only if there exists a quadratic Gröbner basis of the toric ideal of the poset (Theorem 1.2). In order to construct a quadratic Gröbner basis, the most difficult point is to find a suitable monomial order on a polynomial ring. Strong perfect elimination orderings of strongly chordal graphs play an important role in overcoming this difficulty. In Section 2, we apply the results in Section 1 to a toric ring arising from a graph. Given a graph G, let A_G be the vertex–edge incidence matrix of G and let E_n be an identity matrix. It is proved that the toric ideal of the configuration $(2E_n \mid A_G)$ has a quadratic Gröbner basis if and only if G is strongly chordal (Theorem 2.2).

1. A Gröbner basis characterization

In this section, we give the main theorem of this paper and its proof. First we present a useful lemma.