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The union-closed sets conjecture almost holds for almost all random bipartite graphs

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ABSTRACT

Frankl's union-closed sets conjecture states that in every finite union-closed family of sets, not all empty, there is an element in the ground set contained in at least half of the sets. The conjecture has an equivalent formulation in terms of graphs: In every bipartite graph with least one edge, both colour classes contain a vertex belonging to at most half of the maximal stable sets.

We prove that, for every fixed edge-probability, almost every random bipartite graph almost satisfies Frankl's conjecture.

1. Introduction

One of the most basic conjectures in extremal set theory is Frankl's conjecture on union-closed set families. A family $\mathcal{F}$ of sets is *union-closed* if $X \cup Y \in \mathcal{F}$ for all $X, Y \in \mathcal{F}$.

Union-closed sets conjecture. *Let $\mathcal{F} \neq \{\emptyset\}$ be a finite union-closed family of sets. Then there is an $x \in \bigcup_{X \in \mathcal{F}} X$ that lies in at least half of the members of $\mathcal{F}$.*

While Frankl [10] dates the conjecture to 1979, it apparently did not appear in print before 1985, when it was mentioned as an open problem in Rival [19]. Despite being widely known, there is only little substantial progress on the conjecture.

The conjecture has two equivalent formulations, one in terms of lattices and one in terms of graphs. For the latter, let us say that a vertex set S in a graph is *stable* if no two of its vertices are adjacent, and that it is *maximally stable* if, in addition, every vertex outside S has a neighbour in S.

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Conjecture 1 (*Bruhn, Charbit, Schaudt and Telle [4]*). *Let G be a bipartite graph with at least one edge. Then each of the two bipartition classes contains a vertex belonging to at most half of the maximal stable sets.*

We prove a slight weakening of Conjecture 1 for random bipartite graphs. For $\delta > 0$, we say that a bipartite graph *satisfies the union-closed sets conjecture up to* δ if each of its two bipartition classes has a vertex for which the number of maximal stable sets containing it is at most $\frac{1}{2} + \delta$ times the total number of maximal stable sets. A *random bipartite graph* is a graph on bipartition classes of cardinalities m and n, where any two vertices from different classes are independently joined by an edge with probability p. We say that *almost every* random bipartite graph has property P if for every $\varepsilon > 0$ there is an N such that, whenever $m + n \geq N$, the probability that a random bipartite graph on $m + n$ vertices has P is at least $1 - \varepsilon$.

We prove:

Theorem 2. *Let $p \in (0, 1)$ be a fixed edge-probability. For every $\delta > 0$, almost every random bipartite graph satisfies the union-closed sets conjecture up to δ.*

While Frankl's conjecture has attracted quite a lot of interest, a proof seems still out of reach. Indeed, a proof that there is always an element contained in just 1% of the sets would be spectacular. It is unknown whether there is *any* constant factor $\lambda > 0$ such that a union-closed family contains an element in at least λ of the sets. We think that, in this light, the weakening of the conjecture we prove by Theorem 2 is rather marginal.

For a survey of the literature on Frankl's conjecture we refer to [5]. Some of the earliest results verified the conjecture for few sets or few elements in the ground set, that is, when $n = |\mathcal{F}|$ or $m = |\bigcup_{X \in \mathcal{F}} X|$ are small. The current best results show that the conjecture holds for $m \leq 11$, which is due to Bošnjak and Marković [3], and for $n \leq 46$, proved by Lo Faro [12] and independently Roberts and Simpson [20]. The conjecture is also known to be true when n is large compared to m, that is $n \geq 2^m - \frac{1}{2}\sqrt{2^m}$ (Nishimura and Takahashi [15]). The latter result was improved upon by Czédli [6], who shows that $n \geq 2^m - \sqrt{2^m}$ is enough. Recently, Balla, Bollobás and Eccles [1] pushed this to $n \geq \lceil \frac{1}{3} 2^{m+1} \rceil$.

The lattice formulation of the conjecture was apparently known from very early on, as it is already mentioned in Rival [19]. Poonen [16] investigated several variants and gave proofs for geometric as well as distributive lattices. Reinhold [18] extended this, with a very concise argument, to lower semimodular lattices. Finally, the conjecture holds as well for large semimodular lattices and for planar semimodular lattices (Czédli and Schmidt [8]).

The third view, in terms of graphs, on the union-closed sets conjecture is more recent. So far, the graph formulation is only verified for chordal-bipartite graphs, subcubic bipartite graphs, bipartite series–parallel graphs and for bipartitioned circular interval graphs (Bruhn, Charbit, Schaudt and Telle [4]).

One of the main techniques that is used for the set formulation of Frankl's conjecture as well as for the lattice formulation, is averaging: The average frequency of an element is computed, and if that average is at least half of the size of the family, it is concluded that the conjecture holds for the family. Averaging is also our main tool. We discuss averaging and its limits in Section 3.

2. Basic tools and definitions

In our graph-theoretic notation we usually follow Diestel [9], while we refer to Bollobás [2] for more details on random graphs.

All our graphs are finite and simple. We always consider a bipartite graph G to have a fixed bipartition, which we denote by $(L(G), R(G))$. When discussing the bipartition classes, we will often refer to $L(G)$ as the *left side* and to $R(G)$ as the *right side* of the graph.

Throughout the paper we consider a fixed edge probability p with $0 < p < 1$; and we will always put $q = 1 - p$. A *random bipartite graph* G is a bipartite graph where every pair $u \in L(G)$ and $v \in R(G)$ is joined by an edge independently with probability p. We denote by $\mathcal{B}(m, n; p)$ the probability space