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Improved bound on the oriented diameter of graphs with given minimum degree

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ABSTRACT

In 2015, Bau and Dankelmann showed that every bridgeless graph G of order n and minimum degree δ has an orientation of diameter at most $11\frac{n}{\delta+1}+9$. As they were convinced that this bound is not best possible, they posed the problem of improving it.

In this paper, we prove that such a graph G has an orientation of diameter less than $7\frac{n}{\delta+1}$ and give a polynomial-time algorithm to construct one.

1. Introduction and terminology

For terminology not explicitly introduced here, we refer to [1]. We only consider graphs and digraphs without loops and parallel edges/arcs. For a graph G, its vertex set and its edge set are denoted by $V(G)$ and $E(G)$, respectively. Instead of $\{x, y\} \in E(G)$, we write $xy \in E(G)$. Analogous notation is used for the arc set $A(D)$ of a digraph D. An *orientation* D of an undirected graph G is a digraph on the same vertex set $V(G)$ obtained by assigning every edge of G an orientation. By a *suborientation* of a graph G, we mean an orientation of a subgraph of G.

A sequence $P = x_1 \dots x_\ell$ of distinct vertices such that $x_i x_{i+1} \in E(G)$, for all $i \in \{1, \dots, \ell-1\}$, is called an (x_1, x_ℓ)-*path* of length $\ell - 1$ in G. The *distance* from a vertex x to a vertex y is the length of a shortest (x, y)-path. It is considered to be infinite, if there is no such path. The *diameter* of a graph is the maximum distance between two of its vertices. For digraphs, analogous definitions hold.

A graph is *connected*, if it contains an (x, y)-path for every pair $\{x, y\}$ of its vertices. A digraph with said property is called *strong*. A *bridge* is an edge of a connected graph after whose removal the graph is no longer connected. A graph is called *bridgeless*, if it is connected and contains no bridge.

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Let $X \subseteq V(G)$. Then $G[X]$ is the subgraph of G induced by X, i.e. $(X, \{xy \mid x, y \in X, xy \in E(G)\})$. $G-X$ denotes the subgraph $G[V(G)\setminus X]$. For a single vertex x, we also write $G-x$. For a subset $B \subseteq E(G)$ of edges, $G - B$ is defined as $(V(G), E(G) \setminus B)$. This notation extends to digraphs. The *neighbourhood* $N_G(x)$ of a vertex $x \in V(G)$ is the set of all vertices adjacent to x in G and the *closed neighbourhood* $N_G[x]$ is $N_G(x) \cup \{x\}$. We call $|N_G(x)|$ the *degree* of x.

A well-known result, due to Robbins [13], states that an undirected graph admits a strong orientation, if and only if it is bridgeless. This 1939 theorem was inspired by an application in traffic control, to make traffic flow more efficient by use of one-way streets, to be precise. Therefore, the following question emerges naturally: Which strong orientation is best? As concepts of efficiency differ (see, e.g., [14]), obviously, there is more than one answer to this question. For a graph representing an emergency system, for example, one might seek to minimize the distance from any vertex to a specified vertex representing, say, a hospital.

In this paper, our aim is to minimize the maximum distance between any two vertices in an orientation of a given graph. In other words, we consider the *oriented diameter*, which is defined as the minimum diameter of any strong orientation of the undirected graph:

$$\mathrm{diam}_{\mathrm{or}}(G) := \min\{\mathrm{diam}(D) \mid D \text{ is a strong orientation of } G\}.$$

As early as 1978, Chvátal and Thomassen [4] proved the determination of the oriented diameter of a given graph to be $\mathcal{NP}$-complete, justifying the subsequent search for upper bounds on the oriented diameter. Chvátal and Thomassen [4] themselves gave the upper bound $\mathrm{diam}_{\mathrm{or}}(G) \leq 2d^2 + 2d$, where d is the diameter of the undirected bridgeless graph G, but were not able to prove its sharpness. Only a family of graphs with oriented diameter at least $\frac{1}{2}d^2 + d$ is known, guaranteeing a sharp upper bound somewhere in between.

Since then, several bounds in terms of certain graph invariants such as the diameter [5,8,9,12], the radius [3], the domination number [7,11] and the minimum degree [2] have been given. For a survey, due to Koh and Tay, on the numerous further results on the oriented diameter published over the past decades see [10].

In [2], Bau and Dankelmann considered the correlation between the minimum degree of a bridgeless graph and its oriented diameter. Inspired by the result of Erdős, Pach, Pollack and Tuza [6] that the diameter of connected graphs of order n and minimum degree δ is at most $\frac{3n}{\delta+1} + \mathcal{O}(1)$, Bau and Dankelmann set out to find a similar bound $c\frac{3n}{\delta+1} + \mathcal{O}(1)$, for some constant c, on the oriented diameter of bridgeless graphs of order n and minimum degree δ. They were able to prove its validity for $c = 11/3$ and gave an infinite family of graphs with diameter at least $\frac{3n}{\delta+1} + \mathcal{O}(1)$. Convinced that their result was not best possible, Bau and Dankelmann posed the problem of determining the minimum value c, $1 \leq c \leq 11/3$, such that the bound $c\frac{3n}{\delta+1} + \mathcal{O}(1)$ holds.

In this paper, we prove the best constant c to be at most $7/3$. Furthermore, we give a polynomial-time algorithm to construct the corresponding orientation.

2. Results

Theorem 1. *Let $G = (V, E)$ be a bridgeless graph of order n and minimum degree δ. Then*

$$\mathrm{diam}_{\mathrm{or}}(G) < 7\frac{n}{\delta+1}.$$

Proof. For $\delta \leq 6$, the result is trivially true. Thus, let $\delta \geq 7$. For a vertex set $U \subseteq V$ and a vertex $v \in V$, we call v *near* to U, if $v \in U$ or, with respect to G, v either has a neighbour in U or v has two neighbours in common with some vertices of U. Iteratively, we will construct a strong suborientation D' of G with less than $7\frac{n}{\delta+1} - 6$ vertices such that

$$\text{every vertex } v \in V \setminus V(D') \text{ is near to } V(D'). \qquad (*)$$

Before we do so, we consider how such a suborientation implies the result. As the diameter of D' is smaller than $7\frac{n}{\delta+1} - 7$, we then only need to extend D' to an orientation D of G without increasing the diameter by more than 7. We obtain such an extension D by adding short oriented paths from $V(D')$ through $V \setminus V(D')$ back to $V(D')$. We mark the vertices of such paths as processed.