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Ghost series and a motivated proof of the Andrews–Bressoud identities ☆

Shashank Kanade [a,1], James Lepowsky [a], Matthew C. Russell [a], Andrew V. Sills [b]

[a] *Department of Mathematics, Rutgers University, Piscataway, NJ 08854, United States*
[b] *Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460, United States*

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We present what we call a "motivated proof" of the Andrews–Bressoud partition identities for even moduli. A "motivated proof" of the Rogers–Ramanujan identities was given by G.E. Andrews and R.J. Baxter, and this proof was generalized to the odd-moduli case of Gordon's identities by J. Lepowsky and M. Zhu. Recently, a "motivated proof" of the somewhat analogous Göllnitz–Gordon–Andrews identities has been found. In the present work, we introduce "shelves" of formal series incorporating what we call "ghost series," which allow us to pass from one shelf to the next via natural recursions, leading to our motivated proof. We anticipate that these new series will provide insight into the ongoing program of vertex-algebraic categorification of the various "motivated proofs."

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E-mail addresses: skanade@math.rutgers.edu, kanade@ualberta.ca (S. Kanade), lepowsky@math.rutgers.edu (J. Lepowsky), russell2@math.rutgers.edu (M.C. Russell), asills@georgiasouthern.edu (A.V. Sills).

[1] Current address: University of Alberta, Edmonton, Canada.

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1. Introduction

The classical Rogers–Ramanujan partition identities have numerous generalizations in various directions, notable among which are the generalizations by B. Gordon and G.E. Andrews for odd moduli, extended to even moduli by Andrews and D.M. Bressoud. The product sides of the Rogers–Ramanujan identities enumerate the partitions whose parts obey certain restrictions modulo 5, and the sum sides enumerate the partitions with certain difference-two and initial conditions. Generalizations of the Rogers–Ramanujan identities for all odd moduli were discovered by Gordon [33] and Andrews [1] (cf. [4]). Analogous identities for the even moduli of the form $4k + 2$ were discovered by Andrews in [2] and [3] (cf. [4]), and subsequently, for all the even moduli, by Bressoud in [14] (cf. [15]). The Andrews–Bressoud identities state that for any $k \geq 2$ and $i \in \{1, \ldots, k\}$,

$$\frac{\prod_{m\geq 1}(1-q^{2km})(1-q^{2km-k-i+1})(1-q^{2km-k+i-1})}{\prod_{m\geq 1}(1-q^m)} = \sum_{n\geq 0} b_{k,i}(n)q^n,$$

where $b_{k,i}(n)$ is the number of partitions $\pi = (\pi_1, \ldots, \pi_s)$ of n (with $\pi_t \geq \pi_{t+1}$) such that

(1) $\pi_t - \pi_{t+k-1} \geq 2$,
(2) $\pi_t - \pi_{t+k-2} \leq 1$ only if $\pi_t + \pi_{t+1} + \cdots + \pi_{t+k-2} \equiv i + k \pmod 2$,
(3) at most $k - i$ parts of π equal 1.

Here we have replaced r by $k - i + 1$ in the statement of the main theorem of [15]. Also, here and below, q is a formal variable.

The product side above is the generating function for partitions not congruent to 0 or $\pm(k-i+1)$ modulo $2k$, except in the case $i = 1$. The statement of the main theorem in [15] excluded this exceptional case, simply because no natural combinatorial interpretation of the corresponding product side was known at the time, but the proof of the main theorem in [15], in particular, Lemma 3, certainly did cover this case. Building on work of Andrews and R.P. Lewis [8], an elegant combinatorial interpretation of the product in the case $i = 1$ was discovered in [48].

We are motivated by the fact that the Gordon–Andrews–Bressoud identities, as well as more general families of such identities, also arise very naturally from the representation theory of vertex operator algebras, as we will recall below. In [40–42], the Rogers–Ramanujan identities were proved, and the more general Gordon–Andrews–Bressoud identities were interpreted (including the case $i = 1$ for even moduli), using the theory of Z-algebras, which were invented for this very purpose. In [44], this interpretation of the more general Gordon–Andrews–Bressoud identities was strengthened to a vertex-operator-theoretic proof of these identities. The Z-algebra structures came to be understood in retrospect as the natural generating substructures of certain twisted modules for certain generalized vertex operator algebras, once the theory of generalized