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Diagram monoids and Graham–Houghton graphs: Idempotents and generating sets of ideals

James East [a], Robert D. Gray [b,1]

[a] *Centre for Research in Mathematics, School of Computing, Engineering and Mathematics, Western Sydney University, Locked Bag 1797, Penrith, NSW, 2751, Australia*

[b] *School of Mathematics, University of East Anglia, Norwich NR4 7TJ, UK*

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ABSTRACT

We study the ideals of the partition, Brauer, and Jones monoid, establishing various combinatorial results on generating sets and idempotent generating sets via an analysis of their Graham–Houghton graphs. We show that each proper ideal of the partition monoid $\mathcal{P}_n$ is an idempotent generated semigroup, and obtain a formula for the minimal number of elements (and the minimal number of idempotent elements) needed to generate these semigroups. In particular, we show that these two numbers, which are called the rank and idempotent rank (respectively) of the semigroup, are equal to each other, and we characterize the generating sets of this minimal cardinality. We also characterize and enumerate the minimal idempotent generating sets for the largest proper ideal of $\mathcal{P}_n$, which coincides with the singular part of $\mathcal{P}_n$. Analogous results are proved for the ideals of the Brauer and Jones monoids; in each case, the rank and idempotent rank turn out to be equal, and all the minimal generating sets are described. We also show how the rank and idempotent rank results obtained, when applied to the corresponding twisted semigroup algebras (the partition, Brauer, and Temperley–Lieb algebras), allow one to recover formulae for the dimensions of their cell modules (viewed as cellular algebras) which, in the semisimple case, are formulae for the dimensions of the irreducible representations of the algebras. As well as being of algebraic interest, our results relate to sev-

E-mail addresses: J.East@WesternSydney.edu.au (J. East), Robert.D.Gray@uea.ac.uk (R.D. Gray).

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eral well-studied topics in graph theory including the problem of counting perfect matchings (which relates to the problem of computing permanents of $\{0, 1\}$-matrices and the theory of Pfaffian orientations), and the problem of finding factorizations of Johnson graphs. Our results also bring together several well-known number sequences such as Stirling, Bell, Catalan and Fibonacci numbers.

1. Introduction

There has been a lot of interest recently in algebras with a basis consisting of diagrams that are multiplied in some natural diagrammatic way. Examples of such “diagram algebras” include the Brauer algebra [8], Temperley–Lieb algebra [60], and the Jones algebra [90]. All of these examples arise in a natural way as subalgebras of the partition algebra [105], whose basis consists of all set-partitions of a $2n$-element set (see below for a formal definition). The partition algebra first appeared independently in the work of Martin [104,105] and Jones [89]. In both cases, their motivation for studying this algebra was as a generalization of the Temperley–Lieb algebra and the Potts model in statistical mechanics. Since its introduction, the partition algebra has received a great deal of attention in the literature; see for example [47,71,74,76–80,82,92,106,107,109,110,136,138].

All of the diagram algebras mentioned above are examples of cellular algebras, an important class of algebras introduced by Graham and Lehrer in [63]. The fact that these algebras are cellular allows one to obtain information about the semisimplicity of the algebra and about its representation theory, even in the non-semisimple case. In the partition, Brauer, and Temperley–Lieb algebras, the product of two diagram basis elements is always a scalar multiple of another basis element. Using this observation as a starting point, Wilcox [136], showed that these algebras are isomorphic to certain twisted semigroup algebras. By realizing the algebras in this way, many questions concerning the algebras can be related to questions for the corresponding semigroups. For instance, cellularity of the algebra can be deduced from various aspects of the structure of the monoid. The original study of cellular semigroup algebras may be found in [35]; see also [73,74,112,113] for some recent developments. Another example of how the study of these semigroups can give information about the associated algebras may be found in work of the first author [36], who gives presentations for the partition monoid and shows how these presentations give rise to presentations for the partition algebra; see also [37,43,44]. A further example is given in the paper [27] where idempotents in the partition, Brauer and partial Brauer monoids are described and enumerated, and then the results are applied to determine the number of idempotent basis elements in the finite dimensional partition, Brauer and partial Brauer algebras; see also [28].