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Domination in 3-tournaments

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ABSTRACT

A 3-tournament is a complete 3-uniform hypergraph where each edge has a special vertex designated as its tail. A vertex set X dominates T if every vertex not in X is contained in an edge whose tail is in X. The domination number of T is the minimum size of such an X. Generalizing well-known results about usual (graph) tournaments, Gyárfás conjectured that there are 3-tournaments with arbitrarily large domination number, and that this is not the case if any four vertices induce two triples with the same tail. In this short note we solve both problems, proving the first conjecture and refuting the second.

A tournament is an oriented complete graph. The following generalization of tournaments to higher uniformity was suggested by Gyárfás. An r-tournament is a complete r-uniform hypergraph T where each edge has a special vertex designated as its tail. We say that a vertex set X dominates T if every vertex outside X is contained in a hyperedge whose tail is in X. The domination number of T is the minimum size of such a dominating set X. Recently Gyárfás made the following two conjectures about 3-tournaments (see [6]).

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Conjecture 1 *(Gyárfás).*

1. *There are 3-tournaments with arbitrarily large domination number.*
2. *The domination number of a 3-tournament such that any four of its vertices induce at least two edges with the same tail is bounded by a constant.*

These conjectures were motivated by analogous classic results about usual tournaments (see, e.g., [7]). Indeed, it is well known that an n-vertex tournament can have a domination number as large as $(1 + o(1)) \log_2 n$, e.g., random tournaments have this property. On the other hand, if any three vertices of a tournament induce two edges with the same tail, i.e., there are no cyclic triangles, then the tournament is transitive and thus has a dominating set of size 1.

In this short note we construct 3-tournaments of arbitrarily large domination number such that any four vertices induce at least two edges with the same tail. This proves the first conjecture and disproves the second.

The above conjectures turn out to be closely related to a problem about directed graphs. Recall that a directed graph has property S_k if every set of size k is dominated by some other vertex, i.e., for any set X of size k, there is a vertex v such that all k edges between v and X exist and are directed towards X. The girth of a digraph is the minimum length of a directed cycle in it. Myers conjectured in 2003 [8] that every digraph satisfying S_2 has girth bounded by an absolute constant. A similar conjecture was later made in [3], motivated by algorithmic game theory. These conjectures were recently disproved by Anbalagan, Huang, Lovett, Norin, Vetta and Wu [1] (digraphs with property S_2 and girth four were constructed earlier in [2]). Their construction, which is based on a result of Haight [4] (see also [9]) in additive number theory, establishes the following.

Theorem 2 *([1]). For any k and l, there is a directed graph of girth at least l that has property S_k.*

We will use this construction to resolve the above two problems about domination in 3-tournaments. Let D be a digraph of girth at least 4 on a vertex set V, and fix an arbitrary ordering of V. We define T_D to be a 3-tournament on the same set V where the tail of each triple A in T_D is selected as follows. Look at all the directed paths in $D[A]$ of maximum length, and choose the tail of A to be the smallest (according to the ordering we fixed) of the starting vertices. Note that $D[A]$ is acyclic, so this tail has indegree 0 in $D[A]$. The following result together with Theorem 2 proves the existence of 3-tournaments with large domination number, and answers both questions of Gyárfás.

Theorem 3. *If D is a digraph of girth at least 4 with property S_k, then the tournament T_D has domination number at least $k + 1$. Furthermore, if D has girth at least 5, then any four vertices in T_D induce two triples sharing the same tail.*