Journal of Combinatorial Theory, Series A 146 (2017) 169–183

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

The freeness of Ish arrangements

Takuro Abe [a], Daisuke Suyama [b], Shuhei Tsujie [b]

[a] *Institute of Mathematics for Industry, Kyushu University, Fukuoka 819-0395, Japan*
[b] *Department of Mathematics, Hokkaido University, Sapporo, Hokkaido 060-0810, Japan*

ARTICLE INFO

Article history:
Received 18 January 2015
Available online xxxx

Keywords:
Hyperplane arrangement
Ish arrangement
Shi arrangement
Coxeter arrangement
Free arrangement
Supersolvable arrangement
Fiber-type arrangement

ABSTRACT

The Ish arrangement was introduced by Armstrong to give a new interpretation of the q, t-Catalan numbers of Garsia and Haiman. Armstrong and Rhoades showed that there are some striking similarities between the Shi arrangement and the Ish arrangement and posed some problems. One of them is whether the Ish arrangement is a free arrangement or not. In this paper, we verify that the Ish arrangement is supersolvable and hence free. Moreover, we give a necessary and sufficient condition for the deleted Ish arrangement to be free.

1. Introduction

Let $\mathbb{K}$ be a field of characteristic 0 and $\{x_1, \ldots, x_\ell\}$ a basis for the dual space $(\mathbb{K}^\ell)^*$ of the ℓ-dimensional vector space $\mathbb{K}^\ell$. The **Coxeter arrangement** $\mathrm{Cox}(\ell)$ of type $A_{\ell-1}$ (also called the **braid arrangement**) is

$$\mathrm{Cox}(\ell) := \{\{x_i - x_j = 0\} \mid 1 \le i < j \le \ell\},$$

E-mail addresses: abe@imi.kyushu-u.ac.jp (T. Abe), dsuyama@math.sci.hokudai.ac.jp (D. Suyama), tsujie@math.sci.hokudai.ac.jp (S. Tsujie).

http://dx.doi.org/10.1016/j.jcta.2016.09.008
0097-3165/

where $\{x = k\}$ ($x \in (\mathbb{K}^\ell)^*$, $k \in \mathbb{K}$) is the affine hyperplane $\{v \in \mathbb{K}^\ell \mid x(v) = k\}$. Then the **Shi arrangement** $\mathrm{Shi}(\ell)$ and the **Ish arrangement** $\mathrm{Ish}(\ell)$ are defined by

$$\mathrm{Shi}(\ell) := \mathrm{Cox}(\ell) \cup \{\{x_i - x_j = 1\} \mid 1 \leq i < j \leq \ell\},$$
$$\mathrm{Ish}(\ell) := \mathrm{Cox}(\ell) \cup \{\{x_1 - x_j = i\} \mid 1 \leq i < j \leq \ell\}.$$

The Shi arrangement originally defined over $\mathbb{R}$ was introduced by J.Y. Shi [8] in the study of the Kazhdan–Lusztig representation theory of the affine Weyl groups. The Ish arrangement also originally defined over $\mathbb{R}$ was introduced by Armstrong in [1]. He gave a new interpretation of the q, t-Catalan numbers of Garsia and Haiman by using these two arrangements. Armstrong and Rhoades showed that there are some striking similarities between the Shi arrangement and the Ish arrangement in [1,2].

Let $\mathcal{A}$ be an arrangement in $\mathbb{K}^\ell$. Let $L(\mathcal{A})$ be the set of nonempty intersections of hyperplanes in $\mathcal{A}$, which is partially ordered by the reverse inclusion of subspaces. Define the Möbius function $\mu : L(\mathcal{A}) \to \mathbb{Z}$ as follows:

$$\mu(\mathbb{K}^\ell) = 1,$$
$$\mu(X) = -\sum_{\mathbb{K}^\ell \leq Y < X} \mu(Y) \quad (X \neq \mathbb{K}^\ell).$$

Then the **characteristic polynomial** $\chi(\mathcal{A}, t) \in \mathbb{Z}[t]$ of $\mathcal{A}$ is defined by

$$\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}.$$

The following theorem is one of the similarities pointed out by Armstrong.

Theorem 1.1 *([1,5]). The characteristic polynomial of the Shi arrangement and the Ish arrangement are given by*

$$\chi(\mathrm{Shi}(\ell), t) = \chi(\mathrm{Ish}(\ell), t) = t(t - \ell)^{\ell - 1}.$$

Let $\{x_1, \ldots, x_\ell, z\}$ be a basis for V^* of $V := \mathbb{K}^{\ell+1}$. Then, as in [6, Definition 1.15], we have the **cone** $\mathbf{c}(\mathrm{Ish}(\ell))$ over the Ish arrangement which is a central arrangement (namely, an arrangement whose hyperplanes pass through the origin) in V defined by

$$Q\left(\mathbf{c}(\mathrm{Ish}(\ell))\right) = z \prod_{1 \leq i < j \leq \ell} (x_i - x_j)(x_1 - x_j - iz) = 0.$$

Let S be the symmetric algebra of the dual space V^*. S can be identified with the polynomial ring $\mathbb{K}[x_1, \ldots, x_\ell, z]$. Let $\mathrm{Der}(S)$ be the module of derivations of S

$$\mathrm{Der}(S) := \{\theta : S \to S \mid \theta \text{ is } \mathbb{K}\text{-linear},\ \theta(fg) = f\theta(g) + \theta(f)g \text{ for any } f, g \in S\}.$$