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A generalized FKG-inequality for compositions ☆

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ABSTRACT

We prove a Fortuin–Kasteleyn–Ginibre-type inequality for the lattice of compositions of the integer n with at most r parts. As an immediate application we get a wide generalization of the classical Alexandrov–Fenchel inequality for mixed volumes and of Teissier's inequality for mixed covolumes.

1. Introduction

1.1. Consider a finite partially ordered set $(X, \preceq)$ and two non-decreasing (non-negative) functions, $f, g : X \to \mathbb{R}_{\geq 0}$. (Namely, for any $x, y \in X$, if $x \preceq y$ then one

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has $f(x) \le f(y)$ and $g(x) \le g(y)$.) The product function $f \cdot g: \ X \to \mathbb{R}_{\ge 0}$ is also non-decreasing. Take the arithmetic average

$$Av_X(f) := (\sum_{x \in X} f(x))/|X|.$$

A natural question is whether $Av_X(f) \cdot Av_X(g)$ can be compared with $Av_X(f \cdot g)$.

Example 1.1. Suppose that X is totally ordered. Then the non-decreasing functions are just the non-decreasing sequences of real numbers, $0 \le a_1 \le \cdots \le a_n$ and $0 \le b_1 \le \cdots \le b_n$. In this case the comparison of the averages is realized by the classical Chebyshev sum inequality: $(\sum_i a_i)(\sum_j b_j) \le n(\sum_i a_i b_i)$.

On the other hand, if the order on X is not "strong enough" then the inequality utterly fails. Hence, the more precise question is:

$$\text{Which posets does } Av_X(f) \cdot Av_X(g) \le Av_X(f \cdot g) \text{ hold for?} \tag{1}$$

If $(X, \preceq)$ admits an action of some group G, then one can consider the "equivariant" version of this question by taking G-invariant functions f and g.

The fundamental Fortuin–Kasteleyn–Ginibre (FKG) inequality settles the question for a large class of lattices:

Theorem 1.2 *([8], see also [3, pg. 147, Theorem 5]). Let X be a finite distributive lattice. Consider a "measure", $X \overset{\mu}{\to} \mathbb{R}_{\ge 0}$, which is log-supermodular, i.e. $\mu(x \wedge y)\mu(x \vee y) \ge \mu(x)\mu(y)$ for any $x, y \in X$. Then $\Big(\sum_{x \in X} f(x)g(x)\mu(x)\Big) \cdot \sum_{x \in X} \mu(x) \ge \Big(\sum_{x \in X} f(x)\mu(x)\Big) \times \Big(\sum_{x \in X} g(x)\mu(x)\Big)$.*

(The inequality of equation (1) is obtained for the constant measure, $\mu(x) = 1$, which is trivially supermodular.)

One of the interpretation of the FKG inequality is: "in many systems the increasing events are positively correlated" (while an increasing event and a decreasing event are negatively correlated). Hence, the applications of this inequality go far beyond the combinatorics and include e.g. statistical mechanics and probability.

1.2. The condition "X is a distributive lattice" in the above theorem is rather restrictive. Many of the natural posets appearing in arithmetics/algebra/geometry are not of this type. In the current work we establish the inequality of equation (1) for a particular poset $\mathcal{K}_{n,r}$ of ordered compositions, cf. Theorem 3.1. This poset appears frequently in the context of the Young diagrams (representation theory), complete intersections (algebraic geometry), mixed (co)volumes/multiplicities (integral geometry and commutative algebra).