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Evidence for parking conjectures

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ABSTRACT

Let W be an irreducible real reflection group. Armstrong, Reiner, and the author presented a model for parking functions attached to W [3] and made three increasingly strong conjectures about these objects. The author generalized these parking objects and conjectures to the Fuss–Catalan level of generality [26]. Even the weakest of these conjectures would uniformly imply a collection of facts in Coxeter–Catalan theory which are at present understood only in a case-by-case fashion. We prove that when W belongs to any of the infinite families ABCDI, the strongest of these conjectures is generically true.

1. Introduction

The purpose of this paper is to announce evidence supporting a family of conjectures appearing in [3] and [26] related to generalizations of parking functions from the symmetric group $\mathfrak{S}_n$ to an irreducible real reflection group W. Our most important result is that the strongest of these conjectures (the Strong Conjecture) is generically correct whenever W is not of exceptional type. More precisely, we propose a new generic version of the Strong Conjecture (the Generic Strong Conjecture) in all types, and uniformly

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prove that the Generic Strong Conjecture is equivalent to an *a priori* weaker statement (the Intermediate Conjecture of [3,26]). We also prove that the Intermediate Conjecture is true in type A. Let us give some background on and motivation for these conjectures, deferring precise statements of definitions and results to Section 2.

A *(classical) parking function of size* n is a length n sequence $(a_1, \dots, a_n)$ of positive integers whose nondecreasing rearrangement $(b_1 \leq \cdots \leq b_n)$ satisfies $b_i \leq i$ for all $1 \leq i \leq n$. The set Park_n of parking functions of size n carries a natural action of the symmetric group $\mathfrak{S}_n$ given by $w.(a_1, \dots, a_n) := (a_{w(1)}, \dots, a_{w(n)})$ for $w \in \mathfrak{S}_n$ and $(a_1, \dots, a_n) \in \mathsf{Park}_n$. Parking functions were introduced by Konheim and Weiss in computer science [19], but have received a great deal of attention in algebraic combinatorics [4,12,16].

Parking functions have a natural Fuss generalization. Throughout this paper, we fix a choice $k \in \mathbb{Z}_{>0}$ of Fuss parameter. A *(classical) Fuss parking function of size* n is a length n sequence $(a_1, \dots, a_n)$ of positive integers whose nondecreasing rearrangement $(b_1 \leq \cdots \leq b_n)$ satisfies $b_i \leq k(i-1)+1$ for all $1 \leq i \leq n$. The symmetric group $\mathfrak{S}_n$ acts on the set $\mathsf{Park}_n(k)$ of Fuss parking functions by subscript permutation; when $k = 1$ one recovers $\mathsf{Park}_n(1) = \mathsf{Park}_n$.

In [3], Armstrong, Reiner, and the author presented two generalizations, one algebraic and one combinatorial, of parking functions which are attached to any irreducible real reflection group W. Let h be the Coxeter number of W. The algebraic generalization Park_W^{alg} was defined as a certain quotient $\mathbb{C}[V]/(\Theta - \mathbf{x})$ of the coordinate ring $\mathbb{C}[V]$ of the reflection representation V, where $(\Theta - \mathbf{x})$ is an inhomogeneous deformation of an ideal $(\Theta) \subset \mathbb{C}[V]$ arising from a homogeneous system of parameters Θ of degree $h+1$ carrying V^*. The combinatorial generalization Park_W^{NC} was defined using a certain W-analog of noncrossing set partitions [6,24]. The combinatorial model Park_W^{NC} is easier to visualize and has connections with W-noncrossing partitions, but the algebraic model Park_W^{alg} is easier to understand in a type-uniform fashion.

The combinatorial parking space Park_W^{NC} and the algebraic space Park_W^{alg} carry actions of not just the reflection group W, but also the product $W \times \mathbb{Z}_h$ of W with an order h cyclic group. Armstrong, Reiner, and the author made a sequence of conjectures (Weak, Intermediate, and Strong) of increasing strength about this action [3]. We refer to these collectively as the Main Conjecture.

The Weak Conjecture gives a character formula for the (permutation) action of $W \times \mathbb{Z}_h$ on the combinatorial parking space Park_W^{NC}. The Intermediate and Strong Conjectures assert a strong form of isomorphism $\mathsf{Park}_W^{NC} \cong V^\Theta$ between the combinatorial parking space and a "parking locus" V^Θ attached to the algebraic parking space Park_W^{alg}. The Intermediate Conjecture asserts that this isomorphism holds for one particular choice of the "parameter" Θ, whereas the Strong Conjecture asserts that any choice of Θ would give our isomorphism. Even the Weak Conjecture uniformly implies a collection of uniformly stated facts in Coxeter–Catalan theory which are at present only understood in a case-by-case fashion (see Subsection 2.6 for a statement of these facts).

This setup was extended to the Fuss setting in [26]. The k-W-combinatorial and algebraic parking spaces $\mathsf{Park}_W^{NC}(k)$ and $\mathsf{Park}_W^{alg}(k)$ were defined and specialize as