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Maximal-clique partitions and the Roller Coaster Conjecture

Jonathan Cutler [a,1], Luke Pebody [b]

[a] Department of Mathematical Sciences, Montclair State University, Montclair, NJ 07043, USA
[b] Rokos Capital Management LLP, 23 Savile Row, London W1S 2ET, UK

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ABSTRACT

A graph G is *well-covered* if every maximal independent set has the same cardinality q. Let $i_k(G)$ denote the number of independent sets of cardinality k in G. Brown, Dilcher, and Nowakowski conjectured that the independence sequence $(i_0(G), i_1(G), \ldots, i_q(G))$ was unimodal for any well-covered graph G with independence number q. Michael and Traves disproved this conjecture. Instead they posited the so-called "Roller Coaster" Conjecture: that the terms

$$i_{\lceil \frac{q}{2} \rceil}(G), i_{\lceil \frac{q}{2} \rceil+1}(G), \ldots, i_q(G)$$

could be in any specified order for some well-covered graph G with independence number q. Michael and Traves proved the conjecture for $q < 8$ and Matchett extended this to $q < 12$.

In this paper, we prove the Roller Coaster Conjecture using a construction of graphs with a property related to that of having a maximal-clique partition. In particular, we show, for all pairs of integers $0 \leq k < q$ and positive integers m, that there is a well-covered graph G with independence number q for which every independent set of size $k + 1$ is contained in a unique maximal independent set, but each independent set of

E-mail addresses: jonathan.cutler@montclair.edu (J. Cutler), luke@pebody.org (L. Pebody).

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size k is contained in at least m distinct maximal independent sets.

1. Introduction

The behavior of the coefficients of the independence polynomial of graphs in various classes has produced many interesting problems. For a graph G, we let $\mathcal{I}(G)$ be the set of independent sets in G, i.e., $\mathcal{I}(G) = \{I \subseteq V(G) : E(G[I]) = \emptyset\}$. Also, let $\mathcal{I}_k(G) = \{I \in \mathcal{I}(G) : |I| = k\}$ and $i_k(G) = |\mathcal{I}_k(G)|$. The *independence number of* G is given by $\alpha(G) = \max\{k \in \mathbb{N} : i_k(G) > 0\}$. We let the *independence polynomial of* G be the polynomial defined by

$$I(G;x) = \sum_{k=0}^{\alpha(G)} i_k(G)x^k.$$

We refer to $(i_0(G), i_1(G), \ldots, i_{\alpha(G)}(G))$ as the *independence sequence of* G.

Natural questions arise when one considers possible orderings of the coefficients of the independence sequence over various classes of graphs. If one considers the class of all graphs, then Alavi, Malde, Schwenk, and Erdős [1] proved that the coefficients can be ordered in any way apart from $i_0(G) = 1$. In particular, they proved the following. Throughout the paper, we let $[n] = \{1, 2, \ldots, n\}$.

Theorem 1.1 *(Alavi, Malde, Schwenk, Erdős [1]). Given a positive integer q and a permutation π of $[q]$, there is a graph G with $\alpha(G) = q$ such that*

$$i_{\pi(1)}(G) < i_{\pi(2)}(G) < \cdots < i_{\pi(q)}(G).$$

A graph G is said to be *well-covered* if every maximal independent set in G has the same size. Brown, Dilcher, and Nowakowski [5] conjectured that the independence sequence of any well-covered graph is unimodal. This conjecture was disproved by Michael and Traves [8]. However, they were able to show the following.

Theorem 1.2 *(Michael, Traves [8]). The independence sequence of a well-covered graph G with $\alpha(G) = q$ satisfies*

$$\frac{i_0(G)}{\binom{q}{0}} \leq \frac{i_1(G)}{\binom{q}{1}} \leq \cdots \leq \frac{i_q(G)}{\binom{q}{q}}.$$