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A new plethystic symmetric function operator and the rational compositional shuffle conjecture at $t = 1/q$

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ABSTRACT

Our main result here is that the specialization at $t = 1/q$ of the $Q_{km,kn}$ operators studied in Bergeron et al. [2] may be given a very simple plethystic form. This discovery yields elementary and direct derivations of several identities relating these operators at $t = 1/q$ to the Rational Compositional Shuffle conjecture of Bergeron et al. [3]. In particular we show that if m, n and k are positive integers and (m, n) is a coprime pair then

$$q^{\frac{(km-1)(kn-1)+k-1}{2}} Q_{km,kn}(-1)^{kn}\Big|_{t=1/q} = \frac{[k]_q}{[km]_q} e_{km}\left[X\,[km]_q\right]$$

where as customarily, for any integer $s \geq 0$ and indeterminate u we set $[s]_u = 1 + u + \cdots + u^{s-1}$. We also show that the symmetric polynomial on the right hand side is always Schur positive. Moreover, using the Rational Compositional Shuffle conjecture, we derive a precise formula expressing this

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polynomial in terms of Parking Functions in the $km \times kn$ lattice rectangle.

0. Introduction

The specializations at $t = 1/q$ of all the Extended Shuffle Conjectures are still open to this date. What makes this specialization particularly fascinating is that both sides of the stated identities have combinatorial interpretations. Even for the classical Shuffle conjecture, which was recently proved by Carlsson and Mellit [5], this specialization has not been given a simple combinatorial proof. Proving these identities is quite challenging even in the simplest cases. For instance from the Rational Shuffle Conjecture we can easily derive the following identity, for any coprime pair (m, n).

$$\sum_{D\in\mathcal{D}_{m,n}} q^{\mathrm{coarea}(D)+\mathrm{dinv}(D)} = \frac{1}{[m]_q} \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q \tag{0.1}$$

Here the sum is over Dyck paths in the $m \times n$ lattice rectangle, coarea(D) gives the number of lattice squares above the path and dinv(D) is a Dyck path statistic that can also be given a relatively simple geometric construction. The identity obtained by setting $q = 1$ in (0.1) is an immediate consequence of the Cyclic Lemma, which suggests that this classical result may have a natural q-analogue. The investigations that yielded the present results have been directed towards giving a concrete setting to a variety of identities stated or implied in recent work by the Algebraic Geometers, particularly in [4] and [27]. Unfortunately most of this work appears in language that requires considerable algebraic geometrical background. We have been privileged to have had some of these results translated into a language that we could understand by Eugene Gorsky and Andrei Negut. Many of the theorems we prove here have their origin in this algebraic geometrical literature. Our contribution is to provide proofs that are accessible to the algebraic combinatorial audience. We hope that in doing so, the new results we obtain may be conducive to progress in this challenging area of Algebraic Combinatorics.

We will be dealing here with an algebra $\mathcal{A}$ of linear operators acting on the space Λ of symmetric functions in an infinite alphabet $X = \{x_1, x_2, x_3, \ldots\}$ with coefficients in the field $\mathbb{Q}(q,t)$ of rational functions in the two indeterminates q and t. Given a symmetric function $F[X] \in \Lambda$, it will be convenient to denote by "$\underline{F}$" the operator "*multiplication by* $F[X]$." As is customary, we will denote by "$F^\perp$" the operator dual of $\underline{F}$ with respect to the classical Hall scalar product of symmetric functions.

For a coprime pair (m, n) the $Q_{m,n}$ operators have an elementary definition which, as far as we understand, is due to Burban–Schiffmann in [4]. By taking the lattice point