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Stability of Betti numbers under reduction processes: Towards chordality of clutters

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ABSTRACT

For a given clutter $\mathcal{C}$, let $I := I(\bar{\mathcal{C}})$ be the circuit ideal in the polynomial ring S. In this paper, we show that the Betti numbers of I and $I + (\mathbf{x}_F)$ are the same in their non-linear strands, for some suitable $F \in \mathcal{C}$. Motivated by this result, we introduce a class of clutters that we call chordal. This class is a natural extension of the class of chordal graphs and has the nice property that the circuit ideal associated to the complement of any member of this class has a linear resolution over any field. Finally we compare this class with all known families of clutters which generalize the notion of chordality, and show that our class contains several important previously defined classes of chordal clutters.

0. Introduction

Square-free monomial ideals are in strong connection to topology and combinatorics. There are at least two approaches to investigate these ideals in terms of topology or combinatorics. One approach is to associate a simplicial complex to a given square-free

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monomial ideal I, whose faces come from square-free monomials which do not belong to I. Another approach is to associate a clutter to I whose circuits come from the minimal generators of I. The main goal in both cases is to obtain algebraic properties of I via combinatorial or topological properties of associated objects.

One of the highlighted results on this subject is Fröberg's Theorem. R. Fröberg in 1990 showed that, the edge ideal of a graph G has a linear resolution if and only if the complement graph $\bar{G}$ is chordal [13]. In particular, for a square-free monomial ideal generated in degree 2, the problem of having linear resolution depends only on the associated graph, and does not depend on the characteristic of the base field. This is not the case for square-free monomial ideals generated in degree $d > 2$. The ideal associated to a triangulation of the projective plane, is a classical example of a square-free monomial ideal generated in degree 3 whose resolution depends on the characteristic of the base field (see e.g. [20, Section 4]). So it is too much to expect a combinatorial characterization (as in Fröberg's theorem) for arbitrary square-free monomial ideals with linear resolution. However, it is reasonable to ask if one may find such a characterization for (square-free) monomial ideals with linear resolution over any field. It is worthwhile to say that this problem is equivalent (via Alexander duality) to characterization of all simplicial complexes which are Cohen–Macaulay over any field (cf. [11, Theorem 3]). As a partial result on this subject, several generalizations of chordality for clutters (hypergraphs) are defined in [9,14,25,26]. They showed that the ideals associated to their classes of clutters have linear resolution over any field. However, it is not so difficult to give a counterexample for the other direction. On the other hand, the authors in [5] worked in the other direction, showing that a square-free monomial ideal that has linear resolution over any field is associated with a chorded simplicial complex. The definition of chorded is somewhat technical. The authors in [5] give an example of an ideal that is chorded in their sense, but which does not admit a linear resolution over $\mathbb{Z}_2$ [5, Example 7.2]. Thus, while the chorded property is a necessary condition for a linear resolution, it is not sufficient.

The main aim of this paper is twofold. First we show that, for a given square-free monomial ideal I, we may add (remove) some generators to (from) I in such a way that the corresponding non-linear strands do not change under this process. Motivated by this result, we then introduce a class of clutters whose associated ideals have linear resolution over any field. The advantage of this definition is that this class contains other known families of clutters with this property. At the moment, we don't know of any clutter that has linear resolution over any field, but that is not in our class (see Question 1).

The paper is organized as follows: In the first section, we present the background material. This involves some preliminaries on graded modules and Betti numbers together with some basic notions of combinatorics. Then in Section 2, we state one of the main theorems of this paper (Theorem 2.1). Indeed, with the required preparations, we show that, if I is a square-free monomial ideal corresponded to a clutter $\mathcal{C}$ and $F \in \mathcal{C}$ is chosen appropriately, then the ideals $I + (\mathbf{x}_F)$ and I share the same Betti numbers in their non-linear strands. In Section 3, we introduce the class $\mathfrak{C}_d$ of chordal clutters.