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Smallest tetravalent half-arc-transitive graphs with the vertex-stabiliser isomorphic to the dihedral group of order 8 [☆]

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ABSTRACT

A connected graph whose automorphism group acts transitively on the edges and vertices, but not on the set of ordered pairs of adjacent vertices of the graph is called half-arc-transitive. It is well known that the valence of a half-arc-transitive graph is even and at least four. Several infinite families of half-arc-transitive graphs of valence four are known, however, in all except four of the known specimens, the vertex-stabiliser in the automorphism group is abelian. The first example of a half-arc-transitive graph of valence four and with a non-abelian vertex-stabiliser was described in Conder and Marušič (2003) [4]. This example has 10752 vertices and vertex-stabiliser isomorphic to the dihedral group of order 8. In this paper, we show that no such graphs of smaller order exist, thus answering a frequently asked question.

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1. Introduction

Let Γ be a connected finite graph and G a group of automorphisms of Γ. If G acts transitively on the set of *vertices*, *edges* or *arcs* of the graph (an *arc* is an ordered pair of adjacent vertices), then Γ is said to be *G-vertex-transitive*, *G-edge-transitive* or *G-arc-transitive*, respectively. Furthermore, if G acts transitively on the vertices, edges, but not arcs of the graph, then Γ is $(G, \frac{1}{2})$*-arc-transitive*. A graph Γ is $\frac{1}{2}$*-arc-transitive* if it is $(G, \frac{1}{2})$-arc-transitive for $G = \mathrm{Aut}(\Gamma)$.

While graphs that are $(G, \frac{1}{2})$-arc-transitive for some group of automorphisms G are rather easy to find (for example, every cycle is such a graph), $\frac{1}{2}$-arc-transitive graphs are considerably more elusive and the question of their existence has been posed as an open problem by Tutte [28] and answered affirmatively by Bouwer [3] a few years later. Since cycles are arc-transitive and every $(G, \frac{1}{2})$-arc-transitive graph has to have even valence (as observed already in [28]), the smallest admissible valence for a $\frac{1}{2}$-arc-transitive graph is 4; and indeed, the smallest $\frac{1}{2}$-arc-transitive graph is tetravalent and of order 27. The tetravalent $\frac{1}{2}$-arc-transitive graphs (and half-arc-transitive graphs in general) were much studied by many authors from different points of view, ranging from purely combinatorial [7,13,17,25–27,29], geometrical [15], to permutation group theoretical [1, 8,10] and abstract group theoretical [16,20,24].

As observed by Marušič and Nedela [16], a group theoretical result of Glauberman [6] implies that the vertex-stabiliser G_v in a tetravalent $(G, \frac{1}{2})$-arc-transitive graph is a group of order 2^s for some $s \geq 1$, of nilpotency class at most 2, generated by s involutions, and satisfying certain addition conditions (see [16, Theorem 1.1] for details and [20, Theorem 1.2] for a generalisation of this result to graphs of larger valence).

While each of the 2-groups described in [16, Theorem 1.1] can indeed occur as the vertex-stabiliser G_v in a tetravalent $(G, \frac{1}{2})$-arc-transitive graph, it remains an open problem which of the groups of [16, Theorem 1.1] can occur as the vertex-stabiliser in the full automorphism group of a tetravalent $\frac{1}{2}$-arc-transitive graph. This problem was resolved for the case of abelian vertex-stabilisers in [14], where for every positive integer s, a tetravalent $\frac{1}{2}$-arc-transitive graph with G_v isomorphic to $\mathbb{Z}_2^s$ was constructed.

Non-abelian vertex-stabilisers seem to be much more elusive in this respect. The first example of a tetravalent $\frac{1}{2}$-arc-transitive graph with a non-abelian vertex-stabiliser has been constructed by Conder and Marušič [4]. Their example has order 10752 and vertex-stabiliser isomorphic to the dihedral group D_4 of order 8. In [5], two more examples with $\mathrm{Aut}(\Gamma)_v \cong \mathrm{D}_4$ were found (another one of order 10752 and one of order 21870), and also the first known example of a tetravalent $\frac{1}{2}$-arc-transitive graph with a non-abelian vertex-stabiliser of order 16; the latter having order $90 \cdot 3^{10}$. To the best of our knowledge, these four graphs are the only known tetravalent $\frac{1}{2}$-arc-transitive graphs with a non-abelian vertex-stabiliser.

Given that the order of the smallest tetravalent $\frac{1}{2}$-arc-transitive graph with the stabiliser isomorphic to D_4, found by Conder and Marušič more than a decade ago, is rather large, the question of existence of a smaller specimen of this family has often been raised