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On metric properties of maps between Hamming spaces and related graph homomorphisms

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A R T I C L E I N F O

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A B S T R A C T

A mapping of k-bit strings into n-bit strings is called an (α, β)-map if k-bit strings which are more than αk apart are mapped to n-bit strings that are more than βn apart in Hamming distance. This is a relaxation of the classical problem of constructing error-correcting codes, which corresponds to $\alpha = 0$. Existence of an (α, β)-map is equivalent to existence of a graph homomorphism $\bar{H}(k, \alpha k) \to \bar{H}(n, \beta n)$, where $H(n, d)$ is a Hamming graph with vertex set $\{0, 1\}^n$ and edges connecting vertices differing in d or fewer entries.

This paper proves impossibility results on achievable parameters (α, β) in the regime of $n, k \to \infty$ with a fixed ratio $\frac{n}{k} = \rho$. This is done by developing a general criterion for existence of graph-homomorphism based on the semi-definite relaxation of the independence number of a graph (known as the Schrijver's θ-function). The criterion is then evaluated using some known and some new results from coding theory concerning the θ-function of Hamming graphs. As an example, it is shown that if $\beta > 1/2$ and $\frac{n}{k}$ – integer, the $\frac{n}{k}$-fold repetition map achieving $\alpha = \beta$ is asymptotically optimal.

Finally, constraints on configurations of points and hyperplanes in projective spaces over $\mathbb{F}_2$ are derived.

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1. Introduction

Hamming space $\mathbb{F}_2^k$ of binary k-strings, equipped with the Hamming distance is one of the classical objects studied in combinatorics. Its properties that received significant attention are the maximal packing densities, covering numbers, isoperimetric inequalities, list-decoding properties, etc. In this paper we are interested in studying metric properties of maps $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ between Hamming spaces of different dimensions.

Indeed, frequently one is interested in embedding $\mathbb{F}_2^k$ into $\mathbb{F}_2^n$ "expansively", i.e. so that points that were far apart in $\mathbb{F}_2^k$ remain far apart in $\mathbb{F}_2^n$. Two immediate examples of such maps are: the error-correcting codes with rate k/n and minimum distance d satisfy

$$|x - x'| > 0 \implies |f(x) - f(x')| \geq d\,,$$

where here and below $|z| = \|z\|_0 = |\{i : z_i \neq 0\}|$ is the Hamming weight of the vector. Another example is the repetition coding with $f(x)$ mapping x into $\frac{n}{k}$ repetitions of x. This map satisfies:

$$|x - x'| > \alpha k \implies |f(x) - f(x')| > \alpha n\,. \tag{1}$$

With these two examples in mind, we introduce the main concept of this paper.

Definition 1. A map $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ is called an $(\alpha, \beta; k, n)$-map (or simply an (α, β)-map) if αk and βn are integers and for all $x, x' \in \mathbb{F}_2^k$ we have either

$$|f(x) - f(x')| > \beta n \quad \text{or} \quad |x - x'| \leq \alpha k\,, \tag{2}$$

where $\mathbb{F}_2^k$ is the Hamming space of dimension k over the binary field.

We next define the Hamming graphs $H(n, d)$ for integer $d \in [0, n]$ as follows:

$$V(H(n,d)) = \mathbb{F}_2^n, \quad E(H(n,d)) = \{(x, x') : 0 < |x - x'| \leq d\}\,. \tag{3}$$

By $V(G), E(G)$ and $\alpha(G)$ we denote the vertices of G, the edges of G and the cardinality of the maximal independent set of G. All graphs in this paper are simple (without self-loops and multiple edges). By $\bar{G}$ we denote the (simple) graph obtained by complementing $E(G)$ and deleting self-loops.

The relevance of Hamming graphs to this paper comes from the simple observation:

$$\exists(\alpha, \beta; k, n)\text{-map} \quad \iff \quad \bar{H}(k, \alpha k) \to \bar{H}(n, \beta n)\,,$$

where $G \to H$ denotes the existence of a graph homomorphism (see Section 3 for definition).

This paper focuses on proving negative results showing impossibility of certain parameters (α, β). Note that there are a variety of methods that we can use to disprove