Journal of Combinatorial Theory, Series A 145 (2017) 364–368

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

On derivatives of graphon parameters

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ARTICLE INFO

Article history:
Received 4 June 2015
Available online 3 September 2016

Keywords:
Graphons
Graph limits
Graph homomorphisms
Gâteaux derivatives

ABSTRACT

We give a short elementary proof of the main theorem in the paper "Differential calculus on graphon space" by Diao et al. (2015) [2], which says that any graphon parameters whose $(N+1)$-th derivatives all vanish must be a linear combination of homomorphism densities $t(H,-)$ over graphs H on at most N edges.

Let $\mathcal{W} \subset L^\infty([0,1]^2,\mathbb{R})$ denote the set of bounded symmetric measurable functions $f\colon [0,1]^2 \to \mathbb{R}$ (here symmetric means $f(x,y) = f(y,x)$ for all x, y). Let $\mathcal{W}_{[0,1]} \subset \mathcal{W}$ denote those functions in $\mathcal{W}$ taking values in $[0,1]$. Such functions, known as *graphons*, are central to the theory of graph limits [3], an exciting and active research area giving an analytic perspective towards graph theory.

In [2], the authors systematically study the local structure of differentiable graphon parameters. They develop the theory of consistency constraints for multilinear functionals on graphon space, and as a consequence, obtain the result (Theorem 1 below) that is the graphon analog of the following basic fact from calculus: the set of functions whose $(N+1)$-th derivatives all vanish identically is precisely the set of polynomials of degree at most N. For graphons, homomorphism densities $t(H,-)$ play the role of monomials: they

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[1] Y. Zhao was supported by a Microsoft Research PhD Fellowship.

http://dx.doi.org/10.1016/j.jcta.2016.08.007
0097-3165/

generate a ring of smooth functions that separate points and they have the property of vanishing higher derivatives as in Theorem 1. In this short note, we follow a more direct route to prove their result. Our proof avoids the technicalities of the approach in [2].

We begin with some definitions. The space $\mathcal{W}$ is equipped with the *cut norm*

$$\|f\|_\square := \sup_{\text{measurable } S,T\subseteq[0,1]} \left| \int_{S\times T} f(x,y)\,dxdy \right|.$$

Given $g \in \mathcal{W}$, and a measure-preserving map $\phi\colon [0,1] \to [0,1]$, we define $g^\phi(x,y) := g(\phi(x),\phi(y))$. The *cut distance* on $\mathcal{W}$ is defined by $\delta_\square(f,g) := \inf_\phi \left\|f - g^\phi\right\|_\square$ where ϕ ranges over all such measure-preserving maps. Let $\sim$ denote the equivalence relations in $\mathcal{W}$ defined by $f \sim g \Leftrightarrow \delta_\square(f,g) = 0$. It is known that $(\mathcal{W}_{[0,1]}/\sim, \delta_\square)$ is a compact metric space [4].

Functions $F\colon \mathcal{W}_{[0,1]}/\sim \to \mathbb{R}$ are called *class functions* (we import this terminology from [2]; the term *graphon parameter* is also used in the literature). Class functions that are continuous with respect to the cut distance play an important role in graph parameter/property testing [1,5].

Define the *admissible directions at* $f \in \mathcal{W}_{[0,1]}$ as

$$\mathrm{Adm}(f) := \{g \in \mathcal{W} : f + \epsilon g \in \mathcal{W}_{[0,1]} \text{ for some } \epsilon > 0\}.$$

The *Gâteaux derivative of* F *at* $f \in \mathcal{W}_{[0,1]}$ *in the direction* $g \in \mathrm{Adm}(f)$ is defined by (if it exists)

$$dF(f;g) := \lim_{\lambda\to 0^+} \frac{1}{\lambda}(F(f+\lambda g) - F(f)).$$

Higher mixed Gâteaux derivatives are defined iteratively: $d^{N+1}F(f;g_1,\dots,g_{N+1})$ is defined to be the Gâteaux derivative of $d^N F(-;g_1,\dots,g_N)$ at f in the direction g_{N+1}, if this limit exists.

Let $\mathcal{H}_n$ denote the isomorphism classes of multi-graphs with n edges, no isolated vertices, and no self-loops but possible multi-edges. Also let $\mathcal{H}_{\le n} := \bigcup_{j\le n} \mathcal{H}_j$ and $\mathcal{H} := \bigcup_{j\in\mathbb{N}} \mathcal{H}_j$.

For any $H \in \mathcal{H}$, and any $f \in \mathcal{W}$, we define the homomorphism density

$$t(H,f) := \int_{[0,1]^{V(H)}} \prod_{ij\in E(H)} f(x_i,x_j) \prod_{i\in V(H)} dx_i,$$

where $E(H)$ is the multi-set of edges of H. For example, when H consists of two vertices and two parallel edges between them, $t(H,W) = \int_{[0,1]^2} W(x,y)^2\,dxdy$.

Here is the main result of [2].