Journal of Combinatorial Theory, Series A 145 (2017) 400–425

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Multiple chessboard complexes and the colored Tverberg problem

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ARTICLE INFO

Article history:
Received 2 December 2014
Available online 7 September 2016

Keywords:
Tverberg theorem
Multiple chessboard complexes
Shellable complexes
van-Kampen–Flores theorem

ABSTRACT

Following Karaguezian, Reiner and Wachs we study the connectivity degree and shellability of multiple chessboard complexes. Our central new results provide sharp connectivity bounds relevant to applications in Tverberg type problems where multiple points of the same color are permitted. The results presented in this paper also serve as a foundation for the new results of Tverberg–van Kampen–Flores type, as described in the sequel to this paper.

1. An overview and motivation

Chessboard complexes and their generalizations are some of the most studied graph complexes, with numerous applications in and outside combinatorics [1,5,8–10,12,13,15,16,19,27,22,26].

According to Jonsson [12, Chapter 10], the *connectivity degree* of a simplicial complex is one of the five most important and useful parameters in the study of simplicial

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[1] Supported by the Ministry for Science and Technology of Serbia, Grant 174034.

http://dx.doi.org/10.1016/j.jcta.2016.08.008
0097-3165/

complexes of graphs. Following Karaguezian, Reiner and Wachs [13] we study the connectivity degree of *multiple chessboard complexes* (Section 1.4) and their generalizations (Section 2). Our first central result is Theorem 3.2 which improves the 2-dimensional case of [13, Corollary 5.2] and reduces to the 2-dimensional case of [5, Theorem 3.1] in the case of standard chessboard complexes.

Perhaps it is worth emphasizing that our methods allow us to obtain sharp bounds relevant to applications in Tverberg and van Kampen–Flores type problems (see Section 5.1 and [11]). Our results are homotopical in nature and apply to two-dimensional chessboard complexes. For comparison, the focus in [13] is on the computation of homology (with the coefficients in a field) of multidimensional chessboard complexes.

High connectivity degree is sometimes a consequence of the shellability of the complex (or one of its skeletons), see [21] for an early example in the context of chessboard complexes. Our Theorem 4.4 (as the second central result of our paper) provides a sufficient condition for the shellability of multiple chessboard complexes and yields another proof of Theorem 3.2. The construction of the shelling offers a novel point of view on this problem and seems to be new and interesting already in the case of standard chessboard complexes.

Among the initial applications of the new connectivity bounds established by Theorem 3.2 is a result of colored Tverberg type where multiple points of the same color are permitted (Theorem 5.1 in Section 5). After the first version of our paper was submitted to the arXiv we were kindly informed by Günter Ziegler that Theorem 5.1 is implicit in their recent work (see [6], Theorem 4.4 and the remark following the proof of Lemma 4.2).

Other, possibly less standard and more surprising applications of Theorems 3.2 and 4.4 to theorems of Tverberg–van Kampen–Flores type are announced in [11]. These applications provide new evidence that the chessboard complexes and their generalizations are a natural framework for constructing configuration spaces relevant to Tverberg type problems and related problems about finite sets of points in Euclidean spaces.

Caveat: It is customary to visualize a (generalized) chessboard complex in a rectangular $([m] \times [n])$-chessboard. The reader is free to choose either the Cartesian or the matrix enumeration of squares (where $(1, 1)$ is the lower left corner in the former and the upper left corner in the latter case). This should not generally affect the reading of the paper and some care is needed only when interpreting Fig. 1, which assumes the Cartesian point of view.

1.1. Colored Tverberg problems

'Tverberg problems' is a common name for a class of theorems and conjectures about finite sets of points (point clouds) in $\mathbb{R}^d$. In this section we offer a brief introduction into this area of topological combinatorics. For the sake of better readability, in this outline we use (following [25] and [20]) a diagrammatic presentation of these results. The reader is referred to [22], [20, Section 14.4], [26], and [15] for more complete expositions of these problems and the history of the area.