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On judicious partitions of uniform hypergraphs ☆

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A B S T R A C T

Bollobás and Scott showed that the vertices of a graph of m edges can be partitioned into k sets such that each set contains at most $m/k^2 + o(m)$ edges. They conjectured that the vertices of an r-uniform hypergraph, where $r \geq 3$, of m edges may likewise be partitioned into k sets such that each set contains at most $m/k^r + o(m)$ edges. In this paper, we prove the weaker statement that a partition into k sets can be found in which each set contains at most $\frac{m}{(k-1)^r + r^{1/2}(k-1)^{r/2}} + o(m)$ edges. Some partial results on this conjecture are also given.

1. Introduction

An *r-uniform hypergraph* H consists of a finite set $V(H)$ of vertices and a set $E(H)$ of edges, where each edge is an r-subset of $V(H)$. For a vertex $v \in V(H)$, the *degree* $d_H(v)$ (or $d(v)$ for short) of v is the number of edges of H containing v. The *maximum degree* $\Delta(H)$ (or Δ for short) of H is the maximum of $d(v)$ over all vertices v in H. For disjoint subsets X, Y of $V(H)$, we use $E(X)$ to denote the set of edges of H that are contained in X and let $e(X) = |E(X)|$, and $E(X,Y)$ to denote the set of edges of

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H that are contained in $X \cup Y$ and incident with both X and Y and let $e(X,Y) = |E(X,Y)|$.

As usual $[k]$ stands for the set $\{1, 2, \ldots, k\}$.

Graph or hypergraph partitioning problems usually ask for a partition of the vertex set of a graph or hypergraph into pairwise disjoint subsets with various requirements. For example, the well-known Max-Cut Problem asks for a partition V_1, V_2 of the vertex set of a graph G, that maximizes the number of edges with an end in each V_i. It is NP-hard even when restricted to triangle-free cubic graphs [22] and has been a very active research subject in both Combinatorics and Computer Science.

It is well known that every graph of m edges contains a bipartite subgraph with at least $m/2$ edges. Edwards [9,10] proved the essentially best possible result: every graph with m edges has a bipartite subgraph with at least $m/2 + h(m)/4$ edges, where

$$h(m) = \sqrt{2m + \frac{1}{4}} - \frac{1}{2}.$$

Alon [1] showed that Edwards' bound can be increased by $\Omega(m^{1/4})$ for infinitely many m. For special classes of graphs, such as subcubic graphs [20,23], the main term in Edwards' bound can be improved. The Max-Cut Problem for multigraphs and weighted graphs was studied by Alon and Halperin [3] and independently by Bollobás and Scott [8]. Further discussion can be found in [19].

Judicious partitioning problems maximize or minimize several quantities simultaneously. In the Max-Cut setting, the canonical example is the beautiful result of Bollobás and Scott [6]: there is a cut (V_1, V_2) which achieves not only Edwards' bound, but also the best possible bound on $\max\{e(V_1), e(V_2)\}$.

In [6], Bollobás and Scott also considered judicious k-partitions of graphs and proved that every graph G of m edges has a partition $V_1, \ldots, V_k$ such that

$$e(V_i) \le \frac{1}{k^2}m + \frac{k-1}{2k^2}h(m),$$

for each $i \in [k]$.

While there are reasonable bounds for many judicious partitioning problems for graphs [2,11,12,16,21], the analogous problems for hypergraphs seem to be much more difficult. There are some asymptotic results for 3-uniform hypergraphs [13,18], but for r-uniform hypergraphs with $r \ge 4$, there are few results [14]. In 1997, Bollobás and Scott [5] posed the following conjecture.

Conjecture 1. *Let $r \ge 3$ and $k \ge 2$ be fixed integers. Then every r-uniform hypergraph H of m edges has a partition $V_1, \ldots, V_k$ such that*

$$e(V_i) \le \frac{m}{k^r} + o(m),$$

for each $i \in [k]$.