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Ramsey numbers of 3-uniform loose paths and loose cycles ☆

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ABSTRACT

The *3-uniform loose cycle*, denoted by $\mathcal{C}_n^3$, is the hypergraph with vertices $v_1, v_2, \dots, v_{2n}$ and n edges $v_1v_2v_3, v_3v_4v_5, \dots, v_{2n-1}v_{2n}v_1$. Similarly, the *3-uniform loose path* $\mathcal{P}_n^3$ is the hypergraph with vertices $v_1, v_2, \dots, v_{2n+1}$ and n edges $v_1v_2v_3, v_3v_4v_5, \dots, v_{2n-1}v_{2n}v_{2n+1}$. In 2006 Haxell et al. proved that the 2-color Ramsey number of 3-uniform loose cycles on $2n$ vertices is asymptotically $\frac{5n}{2}$. Their proof is based on the method of the Regularity Lemma. Here, without using this method, we generalize their result by determining the exact values of 2-color Ramsey numbers involving loose paths and cycles in 3-uniform hypergraphs. More precisely, we prove that for every $n \geqslant m \geqslant 3$,

$$R\big(\mathcal{P}_n^3, \mathcal{P}_m^3\big) = R\big(\mathcal{P}_n^3, \mathcal{C}_m^3\big) = R\big(\mathcal{C}_n^3, \mathcal{C}_m^3\big) + 1 = 2n + \left\lfloor \frac{m+1}{2} \right\rfloor,$$

and for every $n > m \geqslant 3$, $R(\mathcal{P}_m^3, \mathcal{C}_n^3) = 2n + \lfloor \frac{m-1}{2} \rfloor$. This gives a positive answer to a recent question of Gyárfás and Raeisi.

1. Introduction

For given k-uniform hypergraphs $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_t$ the *Ramsey number* $R(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_t)$ is the smallest number N such that in every t-coloring of the edges of the complete k-uniform hypergraph on N vertices, $\mathcal{K}_N^k$, there is a monochromatic copy of $\mathcal{H}_i$ in color i, for some $1 \leqslant i \leqslant t$. A *k-uniform*

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loose cycle $\mathcal{C}_n^k$ (shortly, a *cycle of length* n) is a hypergraph with the vertex set $\{v_1, v_2, \ldots, v_{n(k-1)}\}$ and with the set of n edges $e_i = \{v_1, v_2, \ldots, v_k\} + (i-1)(k-1)$, $i = 1, 2, \ldots, n$, where we use mod $n(k-1)$ arithmetic and by adding a number t to a set $H = \{v_1, v_2, \ldots, v_k\}$ we mean a shift, i.e., the set which is obtained by adding t to subscripts of each element of H. Similarly, a *k-uniform loose path* $\mathcal{P}_n^k$ (shortly, a *path of length* n) is a hypergraph with vertex set $\{v_1, v_2, \ldots, v_{n(k-1)+1}\}$ and with the set of n edges $e_i = \{v_1, v_2, \ldots, v_k\} + (i-1)(k-1)$, $i = 1, 2, \ldots, n$. For an edge e of a given loose path (also a given loose cycle) $\mathcal{K}$, the first vertex and the last vertex are denoted by $v_{\mathcal{K},e}$ and $\hat{v}_{\mathcal{K},e}$, respectively. For $k = 2$ we get the usual definitions of a cycle C_n and a path P_n with n edges.

Determining the exact values of $R(P_n, P_m)$, $R(P_n, C_m)$ and $R(C_n, C_m)$ are classical results; see [1, 6,5,8,16]. Also, the asymptotic value of $R(C_n, C_n, C_n)$ was obtained by Figaj and Luczak [7]. Moreover, Gyárfás et al. [10] determined the value of $R(P_n, P_n, P_n)$ for sufficiently large n. For a survey, including some results on the Ramsey numbers of paths and cycles, see [15].

There are few known results about the Ramsey numbers of hypergraphs. Recently, this topic has received considerable attention. Haxell et al. [12] proved the following result on the Ramsey number of 3-uniform loose cycles, using the Regularity Lemma.

Theorem 1.1. *For all $\eta > 0$ there exists some $n_0 = n_0(\eta)$ such that for every $n > n_0$, every 2-coloring of $\mathcal{K}^3_{5(1+\eta)n/2}$ contains a monochromatic copy of $\mathcal{C}_n^3$.*

Subsequently, Gyárfás, Sárközy and Szemerédi [11] extended this result to k-uniform loose cycles and proved that for any $k \geqslant 3$ and $\eta > 0$ there exists some $n_0 = n_0(\eta)$ such that every 2-coloring of $\mathcal{K}_N^k$ with $N = (1+\eta)\frac{1}{2}(2k-1)n$ contains a monochromatic copy of $\mathcal{C}_n^k$, i.e., $R(\mathcal{C}_n^k, \mathcal{C}_n^k)$ is asymptotically equal to $\frac{1}{2}(2k-1)n$. All these proofs are based on the hypergraph regularity method. Recently, Gyárfás and Raeisi [9] determined the exact values of 2-color Ramsey numbers of two k-uniform loose triangles and two k-uniform loose quadrangles. They also posed the following question.

Question 1.2. For every $n \geqslant m \geqslant 3$, is it true that

$$R\left(\mathcal{P}_n^3, \mathcal{P}_m^3\right) = R\left(\mathcal{P}_n^3, \mathcal{C}_m^3\right) = R\left(\mathcal{C}_n^3, \mathcal{C}_m^3\right) + 1 = 2n + \left\lfloor \frac{m+1}{2} \right\rfloor ?$$

In particular, is it true that

$$R\left(\mathcal{P}_n^3, \mathcal{P}_n^3\right) = R\left(\mathcal{C}_n^3, \mathcal{C}_n^3\right) + 1 = \left\lceil \frac{5n}{2} \right\rceil ?$$

In connection with Question 1.2, it is known that for every $n \geqslant \lfloor \frac{5m}{4} \rfloor$, $R(\mathcal{P}_n^3, \mathcal{P}_m^3) = 2n + \lfloor \frac{m+1}{2} \rfloor$; see [13]. In this article, we answer Question 1.2 positively. Our proof involves new ideas (though, it modifies certain ideas from [13] at some points), and does not use the Regularity Lemma. Moreover, we show that $R(\mathcal{P}_m^3, \mathcal{C}_n^3) = 2n + \lfloor \frac{m-1}{2} \rfloor$ for any $n > m \geqslant 3$. Indeed, our results yield Theorem 1.1.

The loose paths and loose cycles are examples of hypergraphs with bounded maximum degree. For this class of hypergraphs, it was conjectured that their Ramsey number is linear in terms of their number of vertices. This conjecture has been established by several authors using the hypergraph regularity method; see [3,4,14]. Recently, Conlon, Fox and Sudakov [2] proved this without using the regularity method.

This paper is organized as follows. In Section 2, we state the principal results necessary to prove the main results. In Section 3, we determine the exact value of the Ramsey number of loose cycles in 3-uniform hypergraphs; this generalizes Theorem 1.1. In Section 4, we provide the exact value of the Ramsey number of loose paths in 3-uniform hypergraphs, and finally, in Section 5, the Ramsey number of a loose path and a loose cycle in 3-uniform hypergraphs is determined.

Note. It is shown in [9, Lemma 1] that $(k-1)n + \lfloor \frac{m+1}{2} \rfloor$ is a lower bound for $R(\mathcal{P}_n^k, \mathcal{P}_m^k)$, $R(\mathcal{P}_n^k, \mathcal{C}_m^k)$ and $R(\mathcal{C}_n^k, \mathcal{C}_m^k) + 1$, when $n \geqslant m \geqslant 2$ and $k \geqslant 3$. Here we note that for any $n > m$ and $k \geqslant 3$, $R(\mathcal{P}_m^k, \mathcal{C}_n^k) \geqslant$