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On the chromatic number of a random hypergraph

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ABSTRACT

We consider the problem of k-colouring a random r-uniform hypergraph with n vertices and cn edges, where k, r, c remain constant as $n \to \infty$. Achlioptas and Naor showed that the chromatic number of a random graph in this setting, the case $r = 2$, must have one of two easily computable values as $n \to \infty$. We give a complete generalisation of this result to random uniform hypergraphs.

1. Introduction

We study the problem of k-colouring a random r-uniform hypergraph with n vertices and cn edges, where k, r and c are considered to be constant as $n \to \infty$. We generalise a theorem of Achlioptas and Naor [4] for k-colouring a random graph (2-uniform hypergraph) on n vertices.

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Their theorem specifies the two possible values for the chromatic number of the random graph as $n \to \infty$. We give a complete generalisation of the result of [4]. We broadly follow the approach of Achlioptas and Naor [4], although they rely on simplifications which are available only in the case $r = 2$. We show that these simplifications can be replaced by more general techniques, valid for all $k, r \geq 2$ except $k = r = 2$.

There is an extensive literature on this problem in the case $r = 2$, colouring random graphs. In the setting we consider here, this culminates with the results of Achlioptas and Naor [4], though these do not give a complete answer to the problem. Our results here include those of [4].

There is also a literature for the case $k = 2$, random hypergraph 2-colouring. Achlioptas, Kim, Krivelevich and Tetali [2] gave a constructive approach, but their results were substantially improved by Achlioptas and Moore [3], using non-constructive methods. The results of [3] are asymptotic in r. Our results here include those of [3], but we also give a non-asymptotic treatment. Recently, Coja-Oghlan and Zdeborová [8] have given a small qualitative improvement of the result of [3], which goes beyond what can be proved here. See these papers, and their references, for further information.

Finally, we note that Krivelevich and Sudakov [14] studied a wide range of random hypergraph colouring problems, and some of their results were recently improved by Kupavskii and Shabanov [15]. But, in the setting of this paper, these results are much less precise than those we establish here.

Remark 1.1. After preparing this paper, we learnt of related work by Coja-Oghlan and his coauthors in the case $r = 2$. Coja-Oghlan and Vilenchik [7] improved the upper bound on the k-colourability threshold, restricting the sharp threshold for k-colourability to an interval of constant width, compared with logarithmic width in [4]. (See also Remark 3.7 below.) A small improvement in the lower bound was obtained by Coja-Oghlan [5]. Additionally, Coja-Oghlan, Efthymiou and Hetterich [6] adapted the methods from [7] to study k-colourability of random regular graphs.

1.1. Hypergraphs

Let $[n] = \{1, 2, \ldots, n\}$. Unless otherwise stated, the asymptotic results in this paper are as $n \to \infty$. Consider the set $\Omega(n, r, m)$ of r-uniform hypergraphs on the vertex set $[n]$ with m edges. Such a hypergraph is defined by its edge set $\mathcal{E}$, which consists of m distinct r-subsets of n. Let $N = \binom{n}{r}$ denote the total number of r-subsets.

Now let $\mathcal{G}(n, r, m)$ denote the uniform model of a random r-regular hypergraph with m edges. So $\mathcal{G}(n, r, m)$ consists of the set $\Omega(n, r, m)$ equipped with the uniform probability distribution. We write $G \in \mathcal{G}(n, r, m)$ for a random hypergraph chosen uniformly from $\Omega(n, r, m)$. The edge set $\mathcal{E}$ of this random hypergraph may be viewed as a sample of size m chosen uniformly, without replacement, from the set of N possible edges.

Although our main focus is the uniform model $\mathcal{G}$, it is simpler for many calculations to work with an alternative model. Let $\Omega^*(n, r, m)$ denote the set of all r-uniform multi-