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Right-angled Artin groups on finite subgraphs of disk graphs

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A R T I C L E I N F O

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A B S T R A C T

Koberda proved that if a graph Γ is a full subgraph of a curve graph $\mathcal{C}(S)$ of an orientable surface S, then the right-angled Artin group $A(\Gamma)$ on Γ is a subgroup of the mapping class group $\mathrm{Mod}(S)$ of S. On the other hand, for a sufficiently complicated surface S, Kim–Koberda gave a graph Γ which is not contained in $\mathcal{C}(S)$, but $A(\Gamma)$ is a subgroup of $\mathrm{Mod}(S)$. In this paper, we prove that if Γ is a full subgraph of a disk graph $\mathcal{D}(H)$ of a handlebody H, then $A(\Gamma)$ is a subgroup of the handlebody group $\mathrm{Mod}(H)$ of H. Further, we show that there is a graph Γ which is not contained in some disk graphs, but $A(\Gamma)$ is a subgroup of the corresponding handlebody groups.

1. Introduction

Let $H = H_{g,n}$ be an orientable 3-dimensional handlebody of genus g with n marked points. We regard its boundary ∂H as a compact connected orientable surface $S = S_{g,n}$ of genus g with n marked points. We denote by

$$\xi(H) = \max\{3g - 3 + n, 0\}$$

the *complexity* of H, a measure which coincides with the number of components of a maximal multi-disk in H. We also define the complexity $\xi(S)$ of S as $\xi(S) = \max\{3g - 3 + n, 0\}$. Let Γ be a finite simplicial graph. Through this paper, we denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and the edge set of Γ respectively. The *right-angled Artin group* on Γ is defined by

$$A(\Gamma) = \langle V(\Gamma) \mid [v_i, v_j] = 1 \text{ if and only if } \{v_i, v_j\} \in E(\Gamma)\rangle .$$

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For two groups G_1 and G_2, we write $G_1 \leq G_2$ if there is an embedding from G_1 to G_2, that is, an injective homomorphism from G_1 to G_2. Similarly, we write $\Lambda \leq \Gamma$ for two graphs Γ and Λ if Λ is isomorphic to an induced subgraph of Γ. We denote by $\mathrm{Mod}(H)$ and $\mathrm{Mod}(S)$ the handlebody group of H and the mapping class group of S respectively.

Right-angled Artin groups have attracted much interest from 3-dimensional topology and geometric group theory through the work of Haglund–Wise [3,4] on special cube complexes. In particular, various mathematicians investigate subgroups of right-angled Artin groups or right-angled Artin subgroups of groups. Crisp–Wiest [2] studied surface subgroups of right-angled Artin groups. Kim–Koberda [6] proved that every right-angled Artin group is quasi-isometrically embedded into a right-angled Artin group on an anti-tree, one whose complement graph is a tree.

On the other hand, the geometry of mapping class groups of surfaces is well understood. A handlebody group $\mathrm{Mod}(H)$ of H is a subgroup of the mapping class group $\mathrm{Mod}(S)$ of S. Hamenstädt–Hensel [5] showed that $\mathrm{Mod}(H)$ is exponentially distorted in $\mathrm{Mod}(S)$. Therefore, the geometric properties of handlebody groups may be different from those of mapping class groups. Furthermore, disk graphs are not quasi-isometric to curve graphs (see Masur–Schleimer [9]). Our motivation of this article is whether similar results to the following three theorems hold for handlebody groups and disk graphs:

Theorem 1.1 *([8, Theorem 1.1 and Proposition 7.16]). If $\Gamma \leq \mathcal{C}(S)$, then $A(\Gamma) \leq \mathrm{Mod}(S)$.*

Theorem 1.2 *([7, Theorem 2]). Let S be an orientable surface with $\xi(S) \leq 2$. If $A(\Gamma) \leq \mathrm{Mod}(S)$, then $\Gamma \leq \mathcal{C}(S)$.*

Theorem 1.3 *([7, Theorem 3]). Let S be an orientable surface with $\xi(S) \geq 4$. Then there exists a finite graph Γ such that $A(\Gamma) \leq \mathrm{Mod}(S)$ but $\Gamma \not\leq \mathcal{C}(S)$.*

Definition 1.4. An embedding f from $A(\Gamma)$ to $\mathrm{Mod}(H)$ is *standard* if f satisfies the following two conditions.

(i) The map f maps each vertex of Γ to a multi-disk twist;
(ii) For two distinct vertices u and v of Γ, the support of $f(u)$ is not contained in the support of $f(v)$.

We first prove the following three theorems.

Theorem 1.5. *If $\Gamma \leq \mathcal{D}(H)$, then $A(\Gamma) \leq \mathrm{Mod}(H)$.*

Theorem 1.6. *For any handlebody H with $\xi(H) = 0$ or $\xi(H) = 1$, if $A(\Gamma) \leq \mathrm{Mod}(H)$, then $\Gamma \leq \mathcal{D}(H)$. For any handlebody H with $\xi(H) = 2$, if there exists a standard embedding $f\colon A(\Gamma) \to \mathrm{Mod}(H)$, then $\Gamma \leq \mathcal{D}(H)$.*

Theorem 1.7. *For $H = H_{0,7}$ and $H = H_{1,5}$, there exists a finite graph Γ such that $A(\Gamma) \leq \mathrm{Mod}(H)$ but $\Gamma \not\leq \mathcal{D}(H)$.*

From Theorem 1.7, it follows that the converse of Theorem 1.5 is generally not true. Note that the main technical contribution in this paper is Lemma 5.3 to prove Theorem 1.7. Further, Kim–Koberda proved that having N-thick stars forces the converse of Theorem 1.1.

Theorem 1.8 *([7, Theorem 5]). Suppose S is a surface with $\xi(S) = N$ and Γ is a finite graph with N-thick stars. If $A(\Gamma) \leq \mathrm{Mod}(S)$, then $\Gamma \leq \mathcal{C}(S)$.*

We also prove the following: