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# Every crowded pseudocompact ccc space is resolvable

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ABSTRACT

We prove that every pseudocompact crowed ccc space is $\mathfrak{c}$-resolvable. This gives a partial answer to problems posed by Comfort and García-Ferriera, and Juhász, Soukup and Szentmiklóssy.



## 1. Introduction

*All spaces under discussion are Tychonoff.*

Let $\kappa \geq 2$ be a cardinal. A space $X$ is called *$\kappa$-resolvable* if there is a family of $\kappa$-many pairwise disjoint dense subsets of $X$. By a *resolvable* space we mean a space that is 2-resolvable. Observe that a resolvable space is *crowded*, i.e., has no isolated points. A space is called *irresolvable* if it is not resolvable. The notion of ($\kappa$-)resolvability is due to Hewitt [4] and Ceder [1], respectively. It is known that every locally compact crowded space is $\mathfrak{c}$-resolvable, where $\mathfrak{c}$ denotes the cardinality of the continuum (for details and some historical comments, see Comfort and García-Ferriera [2]). It is also known that there are irresolvable crowded spaces (Hewitt [4]).

Kunen, Szymański and Tall [6] proved assuming $V = L$, that every crowded Baire space is resolvable. Moreover, they showed that if ZFC is consistent with the existence of a measurable cardinal, then ZFC is consistent with the existence of an irresolvable (zero-dimensional) crowded Baire space.

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It was shown in Comfort and García-Ferriera [2, Theorem 6.9] that every countably compact crowded space is $\omega$-resolvable. This result was improved by Pytkeev [8] (see also [5, Theorem 3.6]). He showed that any countably compact crowded space is $\omega_1$-resolvable, and Juhász, Soukup and Szentmiklóssy [5, Theorem 3.6] left open the question whether $\omega_1$ can be improved to $\mathfrak{c}$. It was asked by Comfort and García-Ferriera [2, §7] whether every pseudocompact crowded space is resolvable. Since every pseudocompact space is Baire, the answer is yes if one assumes $V = L$.

We prove here that every pseudocompact crowded space which satisfies the countable chain condition (abbreviated: ccc) is $\mathfrak{c}$-resolvable. This is a partial answer to the aforementioned problems of Comfort and García-Ferriera, and Juhász, Soukup and Szentmiklóssy.

## 2. Preliminaries

A space satisfies the *countable chain condition* (abbreviated: ccc) provided that any family consisting of pairwise disjoint nonempty open subsets of it is countable. A space is *crowded* if it has no isolated points.

**Lemma 2.1.** *Let $X$ be crowded ccc space, and let $W$ be a nonempty open subset of $X$. Then there is a countably infinite family $\mathscr{U}$ of open $F_\sigma$-subsets of $X$ such that*

(1) *for every $U \in \mathscr{U}$, $\overline{U} \subseteq W$,*
(2) *if $U, V \in \mathscr{U}$ are distinct, then $\overline{U} \cap \overline{V} = \emptyset$,*
(3) *$\bigcup \mathscr{U}$ is dense in $W$.*

**Proof.** Pick an arbitrary point $x \in W$. Since $X$ is Tychonoff, its open $F_\sigma$-subsets form a base. Hence we simply let $\mathscr{U}$ be a maximal family of open $F_\sigma$-subsets of $X$ satisfying (1) and (2) and with the additional condition that for every $U \in \mathscr{U}$, $x \notin \overline{U}$. Then (3) follows by maximality, and $\mathscr{U}$ is countable by ccc. It is clear that $\mathscr{U}$ is infinite since $X$ is crowded. □

It is a well-known result of Souslin that every uncountable completely metrizable separable space contains a copy of the Cantor set $2^\omega$ [7, p. 437].

A space is *pseudocompact* if every real valued continuous function on $X$ is bounded. A subspace $Y$ of $X$ is called *$G_\delta$-dense in* $X$ provided that every nonempty $G_\delta$-subset of $X$ meets $Y$. A useful characterization of pseudocompactness was obtained by Gillman and Jerison [3, p. 95, 6I.1]. They showed that a space $X$ is pseudocompact if and only $X$ is $G_\delta$-dense in $\beta X$. Here $\beta X$ denotes the Čech–Stone-compactification of $X$. If $X$ is a space, then a subset $Z$ of $X$ is called a *zero-set* of $X$ if there is a continuous function $f\colon X \to [0, 1]$ such that $f^{-1}(\{0\}) = Z$. See [3] for more information on these concepts.

**Lemma 2.2.** *Let $X$ be pseudocompact space and let $Z$ be a zero-set of $X$. Then $\mathrm{cl}_{\beta X}(Z)$ is a zero-set of $\beta X$.*

**Proof.** This is Gillman and Jerison [3, 8B.5]. □

**Corollary 2.3.** *Let $X$ be pseudocompact and let $Z$ be a zero-set of $X$. If $f\colon X \to K$ is continuous, and $K$ is metrizable, then $f(Z)$ is compact.*

**Proof.** Observe that $f(X)$ is compact, so the function $f$ extends (uniquely) to a continuous function $\beta f\colon \beta X \to K$. Pick an arbitrary $x \in \overline{f(Z)}$. Consider the set $(\beta f)^{-1}(\{x\})$. It is a compact $G_\delta$-subset of $\beta X$. If $(\beta f)^{-1}(\{x\}) \cap \mathrm{cl}_{\beta X}(Z) = \emptyset$, then $x \notin \beta f(\mathrm{cl}_{\beta X}(Z)) \subseteq \overline{f(Z)}$, which is a contradiction. Since $X$ is $G_\delta$-dense in $\beta X$, it consequently follows by Lemma 2.2 that

$$\emptyset \neq ((\beta f)^{-1}(\{x\}) \cap \mathrm{cl}_{\beta X}(Z)) \cap X = f^{-1}(\{x\}) \cap Z,$$

hence $x \in f(Z)$. □