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On the arithmetic of density ☆

Menachem Kojman

Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Be'er Sheva, 84105 Israel

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ABSTRACT

The κ-density of a cardinal $\mu \geq \kappa$ is the least cardinality of a dense collection of κ-subsets of μ and is denoted by $\mathcal{D}(\mu, \kappa)$. The *Singular Density Hypothesis* (SDH) for a singular cardinal μ of cofinality $\mathrm{cf}\mu = \kappa$ is the equation $\overline{\mathcal{D}}(\mu, \kappa) = \mu^+$, where $\overline{\mathcal{D}}(\mu, \kappa)$ is the density of all unbounded subsets of μ of ordertype κ. The *Generalized Density Hypothesis* (GDH) for μ and λ such that $\lambda \leq \mu$ is:

$$\mathcal{D}(\mu, \lambda) = \begin{cases} \mu & \text{if } \mathrm{cf}\mu \neq \mathrm{cf}\lambda \\ \mu^+ & \text{if } \mathrm{cf}\mu = \mathrm{cf}\lambda. \end{cases}$$

Density is shown to satisfy Silver's theorem. The most important case is:

Theorem *(Theorem 2.6). If $\kappa = \mathrm{cf}\kappa < \theta = \mathrm{cf}\mu < \mu$ and the set of cardinals $\lambda < \mu$ of cofinality κ that satisfy the SDH is stationary in μ then the SDH holds at μ.*

A more general version is given in Theorem 2.8.
A corollary of Theorem 2.6 is:

Theorem *(Theorem 3.5). If the Singular Density Hypothesis holds for all sufficiently large singular cardinals of some fixed cofinality κ, then for all cardinals λ with $\mathrm{cf}\lambda \geq k$, for all sufficiently large μ, the GDH holds.*

1. Introduction

Eventual regularity is a recurring theme in cardinal arithmetic since the discovery of pcf theory. Arithmetic rules that do not necessarily hold for all cardinals, can sometimes be seen to hold in appropriate end-segments of the cardinals.

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E-mail address: kojman@math.bgu.ac.il.

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The most famous precursor of modern cardinal arithmetic is *Silver's theorem* [18], which says that if one of the arithmetic equations (1) the *Singular Cardinal Hypothesis* (SCH); or (2) the *Generalized Continuum Hypothesis* (GCH), holds sufficiently often below a singular of uncountable cofinality, then it holds at the singular itself.

Silver's theorem came as a surprise in 1973, shortly after Solovay and Easton employed Forcing, that was discovered by Cohen in 1963, to prove that no non-trivial bound on the power of a regular cardinal could be deduced from information about the powers of smaller cardinals. At the time, all set theorists believed that no such implications existed and that further development of Forcing would clear the missing singular case soon (see [9] for the history of the subject and for a survey of other precursors of pcf theory, e.g. in topology).

The present article concerns the eventual regularity of the cardinal arithmetical function *density*. The density function $\mathcal{D}(\mu,\kappa)$ is defined for cardinals $\kappa \le \mu$ as the least cardinality of a collection $\mathcal{D} \subseteq [\mu]^\kappa$ which is *dense* in $\langle [\mu]^\kappa, \subseteq \rangle$.

A detailed definition and basic properties of density appear in Section 2 below. Let us point out now, though, to one crucial difference between $\mathcal{D}(\mu,\kappa)$ and the exponentiation μ^κ: the function $\mathcal{D}(\mu,\kappa)$ is *not* monotone increasing in the second variable. For example, if μ is a strong limit cardinal of cofinality ω then $\mathcal{D}(\mu,\aleph_0) = \mu^+ > \mathcal{D}(\mu,\aleph_1) = \mu$.

Recently, asymptotic results in infinite graph theory and in the combinatorics of families of sets [10,11] — some of which were proven earlier with the GCH or with forms of the SCH [5,3,7,12,13] — were proved in ZFC by making use of an eventual regularity property of density: that density satisfies a version of Shelah's RGCH theorem. See also [17] on the question whether the use of RGCH in [10] is necessary.

1.1. The results

Three theorems about the eventual behaviour of density are proved below. Theorem 2.6 is a density version of the most popular case of Silver's theorem and Theorem 2.8 is a density version of the general Silver theorem. They deal with the way the behaviour of density at singular cardinals of cofinality κ below a singular μ of cofinality $\theta > \kappa$ bounds the θ-density at μ.

An elementary proof of Silver's theorem [2] was published by Baumgartner and Prikry soon after Silver's original proof was found. An almost identical elementary proof of Silver's theorem was discovered independently of [2] by Jensen in 1973, but was only circulated and not published (see the introduction to [2] and [9]).

Theorem 3.5 states that if the SDH holds eventually at some fixed cofinality κ then the GDH holds for all sufficiently large cardinals μ and $\lambda \le \mu$ such that $\mathrm{cf}\lambda \ge \kappa$. The proof is by induction, and employs Theorem 2.6 in the critical cases.

1.2. Notation and prerequisites

The notation used here is standard in set theory. In particular, the word *cardinal*, if not explicitly stated otherwise, is to be understood as "infinite cardinal". The variables $\kappa, \theta, \mu, \lambda$ stand for infinite cardinals and $\alpha, \beta, \gamma, \delta, i, j$ denote ordinals. By $\mathrm{cf}\mu$ the *cofinality* of μ is denoted. For $\kappa < \mu$ the symbol $[\mu]^\kappa$ denotes the set of all subsets of μ whose cardinality is κ. If $\kappa = \mathrm{cf}\kappa < \mathrm{cf}\mu$ then we denote by S^μ_λ the set of all ordinals below μ whose cofinality is κ. For a set of ordinals X, the set $\mathrm{acc}(X)$ denotes the set of all accumulation points of X.

We assume familiarity with the basics of stationary sets and the non-stationary ideal and acquaintance with Fodor's pressing down theorem. This material is available in every standard set-theory textbook.