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On discretely generated box products

Hector Alonzo Barriga-Acosta [*,1], Fernando Hernández-Hernández [2]

Posgrado Conjunto en Ciencias Matemáticas, UNAM-UMSNH, Morelia, Mich, Mexico

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A topological space X is called *discretely generated* if for any $A \subseteq X$ and $x \in \overline{A}$ there exists a discrete set $D \subseteq A$ such that $x \in \overline{D}$. We solve the Problems 3.19 and 3.3 in [2]. Problem 3.19: Does the space $\{\xi\} \cup \omega$ embed into a box product of real lines when $\xi \in \beta\omega \setminus \omega$? For any $\xi \in \beta\omega \setminus \omega$, we answer negatively. Problem 3.3: Is any box product of first countable spaces discretely generated? We answer positively by assuming that the spaces are regular.

1. Introduction

Many results concerning to discretely generated spaces have been shown in [1] and [2]. Theorem 2.6 [2] by V. Tkachuk and R. Wilson says that if X_t is a monotonically normal space, then the box product $\square_{t \in T} X_t$ is discretely generated. Hence, the spaces $\square\mathbb{R}^\kappa$, $\square(\omega+1)^\kappa$ and $\square(\{\xi\} \cup \omega)^\kappa$ are discretely generated, for any cardinal κ.

Let $\mathcal{V}$ be the countable regular maximal space due to Eric van Douwen [3]. It was shown in [1] that $\mathcal{V}$ is not discretely generated. Since $\square\mathbb{R}^\kappa$ is discretely generated and this property is hereditary, there is no embedding from $\mathcal{V}$ to $\square\mathbb{R}^\kappa$. The authors of [2] then wondered if there were more countable regular spaces that do not embed into a box product of real lines, that is the motivation of Problem 3.19 in [2]. We generalize their Example 2.10, part b).

* Corresponding author.
E-mail addresses: hector.alonsus@gmail.com (H.A. Barriga-Acosta), fhernandez@fismat.umich.mx (F. Hernández-Hernández).

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A space X is called *monotonically normal* if for every $U \in \tau(X)$ and $x \in U$ there is a set $O(x,U) \in \tau(x,X)$ such that $O(x,U) \cap O(y,V) = \emptyset$ implies $x \in V$ or $y \in U$. Of course, being monotonically normal implies normality. Every metric space is monotonically normal. However, there is no relation between being first countable and monotonically normal. For example, $\{\xi\} \cup \omega$ is a monotonically normal non first countable space. On the other hand, it is well known that the square of the Sorgenfrey line $\mathbb{R}_l^2$ is a regular first countable non normal space, and thus, non monotonically normal. However, the space $\square\mathbb{R}_l^\omega$ is discretely generated by our result.

2. Strategy, notation and terminology

We use standard terminology and follow Engelking [4]. All spaces we consider are assumed to be Hausdorff. If X is a space then $\tau(X)$ is its topology. If X_t is a topological space for every $t \in T$, then the *box product* $\square_{t\in T} X_t$ is the set-theoretic product $\prod_{t\in T} X_t$ with the topology generated by the family $\{\prod_{t\in T} U_t : U_t \in \tau(X_t)\}$. The set of natural numbers is denoted by ω and we use the symbol $\mathbb{R}$ for the real line with its usual topology.

A space X is *discretely generated at a point* $x \in X$ if for any $A \subseteq X$ with $x \in \overline{A}$ there exists a discrete set $D \subseteq A$ such that $x \in \overline{D}$. The space X is *discretely generated* if it is discretely generated at every point $x \in X$.

Let X be a set, $A \subseteq X^\kappa$, κ a cardinal, $S \subseteq \kappa$ and $b \in X^\kappa$. We denote the *support of* $a \in X^\kappa$ *respect to* b by $supp_b(a) = \{\alpha \in \kappa : a(\alpha) \neq b(\alpha)\}$. The *restriction of* a *to* S is the element $a \restriction S \in X^S$ defined as $(a \restriction S)(s) = a(s)$, as well as $A_{S,b} = \{a \in A : supp_b(a) = S\}$ and $A \restriction S = \{a \restriction S \in X^S : a \in A\}$. We denote by $\overrightarrow{\omega}$ the element in $\square(\omega+1)^\omega$ such that for every $n \in \omega$, $\overrightarrow{\omega}(n) = \omega$. When we talk about the "support" in $\square(\omega+1)^\omega$, we use $supp(a)$ instead of $supp_{\overrightarrow{\omega}}(a)$ and A_S instead of $A_{S,\overrightarrow{\omega}}$.

Also, given a function $h \in \omega^\omega$ and an element $a \in \square(\omega+1)^\omega$, we define the *neighborhood of* a *by* h to be the set of the form

$$N_h(a) = \square\{\{a(n)\} : n \in supp(a)\} \times \square\{(h(n), \omega] : n \in \omega \setminus supp(a)\}.$$

Finally, we recall the following definitions on ω^ω: For $f, g \in \omega^\omega$, define $f \leq^* g$ iff $\exists n \in \omega\ \forall m \geq n\ (f(m) \leq g(m))$. A family $\mathcal{F} \subseteq \omega^\omega$ is $\leq^*$-*bounded* if $\exists g \in \omega^\omega\ \forall f \in \mathcal{F}\ (f \leq^* g)$. A family $\mathcal{F} \subseteq \omega^\omega$ is $\leq^*$-*dominant* if $\forall g \in \omega^\omega\ \exists f \in \mathcal{F}\ (g \leq^* f)$.

- $\mathfrak{b} = min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega$ is not $\leq^*$-bounded$\}$
- $\mathfrak{d} = min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega$ is $\leq^*$-dominant$\}$

3. Facts and some definitions

Let $\beta\omega$ denote the Stone–Čech compactification of ω. If $\xi \in \beta\omega \setminus \omega$, then $\{\xi\} \cup \omega$ inherits the subspace topology of $\beta\omega$.

Remark 1. Let $\xi \in \beta\omega \setminus \omega$, then we have the following for the space $\{\xi\} \cup \omega$:

1. $U \in \xi$ if and only if $\xi \in \overline{U}$.
2. If $\xi \in \overline{U} \cap \overline{V}$, then $U \cap V \neq \emptyset$.

Lemma 2. *If a set* $A \subseteq \square(\omega+1)^\omega$ *satisfies* $\forall a \in A\ (|supp(a)| = \omega)$ *and has size less than* $\mathfrak{b}$*, then* $\overrightarrow{\omega} \notin \overline{A}$*.*