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Inverse limits with bonding functions whose graphs are connected

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ABSTRACT

First, answering a question by Roškarič and Tratnik, we present inverse sequences of simple triods or simple closed curves with set-valued bonding functions whose graphs are arcs and the limits are n-point sets. Second, we present a wide class of zero-dimensional spaces that can be obtained as the inverse limits of arcs with one set-valued function whose graph is an arc.

1. Introduction

The inverse limits with upper semicontinuous bonding functions were introduced by Mahavier in [4]. Since then, they became a very popular subject of investigation, especially in the case when all the factor spaces are arcs. Then a book [3] by Ingram and Mahavier (one of a very few in continuum theory) containing a lot of information about the subject was published and the subject became even more popular.

It is known that the inverse limit of compact nonempty spaces with upper semicontinuous functions is nonempty (see [2]), but in some cases it can be degenerate even if the factor spaces are not. In [1] I. Banič and J. Kennedy, and in [6] Roškarič and Tratnik independently showed that if f is an upper semicontinuous function whose graph is connected, then $\varprojlim\{[0,1], f\}$ is either degenerate or infinite. Here we show by counterexamples that this theorem is no longer true if we replace $[0,1]$ by a circle or by a triod. Moreover, we present a wide class of zero-dimensional spaces that can be obtained as the inverse limits of arcs with one set-valued function. At the end of the two sections some open problems are asked.

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2. Preliminaries

In this article we consider metric spaces only. A *continuum* is a nonempty, compact and connected metric space.

If X is a continuum, then 2^X denotes the family of all nonempty closed subsets of X.

The *graph* $G(f)$ of a function $f : X \to 2^Y$ is the set of all points $\langle x, y\rangle \in X \times Y$ such that $y \in f(x)$.

Given compact metric spaces X and Y, a function $f : X \to 2^Y$ is *upper semicontinuous* if for each open set $V \subset Y$ the set $\{x \in X | f(x) \subset V\}$ is a an open set in X. It is known that a function between compact spaces is upper semicontinuous if and only if its graph is closed.

If $\{X_i : i \in \{1, 2, \dots\}\}$ is a countable collection of compact metric spaces each with a metric d_i bounded by 1, then $\Pi_{i=1}^{\infty} X_i$ is the countable product of the collection $\{X_i : i \in \{1, 2, \dots\}\}$ with the metric given by $d(\langle x_1, x_2, \dots\rangle, \langle y_1, y_2, \dots\rangle) = \sum_{i=1}^{\infty} \frac{d_i(x_i,y_i)}{2^i}$. For each j, let $\pi_j \colon \Pi_{i=1}^{\infty} X_i \to X_j$ be defined by $\pi_j\big(\langle x_1, x_2, \dots\rangle\big) = x_j$ that is, π_j is the projection map onto the j-th factor space. For each i let $f_i \colon X_{i+1} \to 2^{X_i}$ be a set valued function where 2^{X_i} denotes the hyperspace of all nonempty closed subsets of X_i. The inverse limit of the sequence of pairs $\{(X_i, f_i)\}$, denoted by $\varprojlim\{X_i, f_i\}$, is defined to be the set of all points $\langle x_1, x_2, \dots\rangle$ in $\Pi_{i=1}^{\infty} X_i$ such that $x_i \in f_i(x_{i+1})$. The functions f_i are called *bonding functions*. For a finite sequence $\mathbf{x} = \langle x_1, x_2, \dots, x_n\rangle$ and finite or infinite sequence $\mathbf{y} = \langle y_1, y_2, \dots\rangle$, let $\mathbf{x} \oplus \mathbf{y} = \langle x_1, x_2, \dots, x_n, y_1, y_2, \dots\rangle$. More information on inverse limits with upper semicontinuous bonding functions can be found for example in the book [3].

A continuum X is called a *dendrite* if it is locally connected and it contains no simple closed curves.

A continuum X is a *hereditarily unicoherent* if for any two subcontinua A and B of X the intersection $A \cap B$ is connected. Consequently, by induction, the intersection of any finite family of subcontinua of X is connected, and since X is compact the intersection of any family of subcontinua of X is connected. As a consequence, for any subset S of X there is a unique continuum C such that $S \subset C$ and C is contained in any continuum that contains S. Here C is the intersection of all continua that contain S. The continuum C is called the *irreducible continuum containing* S.

3. Counterexamples

Theorem 1 below has been proved by Banič and Kennedy in [1] and also by Roškarič and Tratnik in [6] independently. In this section, we show that, this theorem cannot be generalized further by replacing $[0, 1]$ by a circle nor by a simple triod. We provide examples of inverse sequences of circles or of simple triods with set-valued bonding functions whose graphs are arcs and the limits are n-point sets.

Theorem 1. (Banič and Kennedy, Roškarič and Tratnik) *Suppose that $f : [0, 1] \to 2^{[0,1]}$ is an upper semicontinuous function whose graph $G(f)$ is connected. Then $\varprojlim\{[0, 1], f\}$ consists of either one or infinitely many points.*

Definition 2. Let X be a compact metric space. For an upper semicontinuous (not necessarily surjective) function f on X, and a positive integer n, define $P_n(f) = \{x \in X :$ there is $x_n \in X$ such that $\langle x_n, x\rangle \in G(f^n)\}$, and let $P(f) = \bigcap_{n=1}^{\infty} P_n(f)$.

Note that $f|_{P(f)} : P(f) \to 2^{P(f)}$ is surjective and that if f is surjective, then $P(f) = X$.

The following theorem is a generalization of Theorem 3.4 of [1].

Theorem 3. *Suppose X is a compact metric space and $f : X \to 2^X$ is upper semicontinuous. Then $\varprojlim\{X, f\} = \varprojlim\{P(f), f|_{P(f)}\}$.*