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On a nontrivial knot projection under (1, 3) homotopy

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A B S T R A C T

In 2001, Östlund formulated the question: are Reidemeister moves of types 1 and 3 sufficient to describe a homotopy from any generic immersion of a circle in a two-dimensional plane to an embedding of the circle? The positive answer to this question was treated as a conjecture (Östlund conjecture). In 2014, Hagge and Yazinski disproved the conjecture by showing the first counterexample with a minimal crossing number of 16. This example is naturally extended to counterexamples with given even minimal crossing numbers more than 14. This paper obtains the first counterexample with a minimal crossing number of 15. This example is naturally extended to counterexamples with given odd minimal crossing numbers more than 13.

1. Introduction

A *knot projection* is the image of a generic immersion of a circle into a 2-sphere. Thus, any self-intersection of a knot projection is a transverse double point, which is simply called a *double point*. A *trivial knot projection* is a knot projection with no double points. Any two knot projections are related by a finite sequence of Reidemeister moves of types 1, 2, and 3. Here, Reidemeister moves of types 1, 2, and 3, denoted by RI, RII, and RIII, are defined in Fig. 1. If two knot projections P and P' are related by a finite sequence generated by RI and RIII, then P and P' are *(1, 3) homotopic*. The relation becomes an equivalence relation and is called *(1, 3) homotopy*.

In 2001 [2], Östlund formulated a question as follows:

Östlund Question. Are Reidemeister moves RI and RIII sufficient to obtain a homotopy from any generic immersion $S^1 \to \mathbb{R}^2$ to an embedding?

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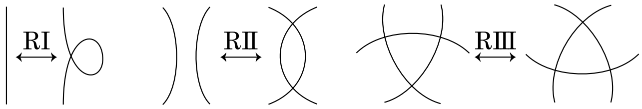

Fig. 1. RI, RII, and RIII.

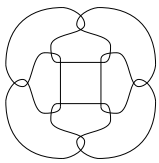

Fig. 2. Hagge–Yazinski's example P_{HY}.

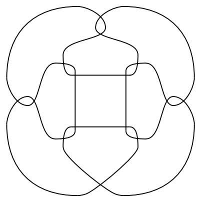

Fig. 3. A knot projection.

In [1], Hagge and Yazinski obtained an answer to this question as follows ([1] treated the positive answer to Östlund's question as "his conjecture", and thus, we call it *Östlund Conjecture*):

Hagge–Yazinski Theorem. *A homotopy from the trivial knot projection to P_{HY} that appears as Fig. 2 cannot be obtained by a finite sequence generated by Reidemeister moves* RI *and* RIII*.*

For a given knot projection P, let $c_{\min}(P) = \min\{$ the number of double points of $P' \mid P'$ and P are related by a finite sequence generated by RI and RIII $\}$. We call $c_{\min}(P)$ the *minimal crossing number* of P.

As a step further, it is easy to find a vast generalization of the Hagge–Yazinski Theorem; there exists an infinite family of counterexamples of the Östlund Conjecture, i.e., non-trivial knot projections under (1, 3) homotopy (Remark 1). However, every knot projection of this family has an even minimal crossing number.

In this paper, we find a counterexample of the conjecture with an odd minimal crossing number. We show the non-triviality of the example by using Hagge–Yazinski techniques [1]. This example is naturally extended to an infinite family of knot projections, each of which is a knot projection with a given odd minimal crossing number. Similar to Remark 1, Theorem 1 naturally implies a knot projection P such that $c_{\min}(P) = 2i + 13$ ($i \in \mathbb{Z}_{>0}$).

Theorem 1. *The knot projection, shown in Fig. 3, and the trivial knot projection cannot be (1, 3) homotopic.*

2. Proof of Theorem 1

As shown in Fig. 4, any P_0 can be decomposed into seven boxes and arcs with no double points. Each box is equivalent to $[0, 1] \times [0, 1]$ under sphere isotopy and in each box, there is a part of a knot projection, called (3, 3)-*tangle* (cf. Definition 1).

Definition 1 *((s, t)-tangle).* Let s and t be positive integers such that $s + t$ is even. An unoriented (s, t)-tangle is the image of a generic immersion of $(s + t)/2$ arcs into $[0, 1] \times [0, 1]$ such that: