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Strong surjections from two-complexes with trivial top-cohomology onto the torus

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ABSTRACT

We build a countable collection of two-dimensional CW complexes with trivial second integer cohomology group and, from each of them, a strong surjection onto the torus. Furthermore, we prove that such two-complexes are the simplest with these properties. This answers, for dimension two, a problem originally proposed in the 2000's for dimension three.

1. Introduction

Let K be a finite and connected two-dimensional CW complex (a two-complex, to shorten). By the Hopf–Whitney Classification Theorem [12, Corollary 6.19 on p. 244], if the top-cohomology group $H^2(K)$ is trivial (integer coefficients is subtended), then all maps from K into the two-sphere are homotopic to a non-surjective map, but we do not know about maps into the others closed surfaces. This problem becomes more interesting when we note that a two-complex K with $H^2(K) = 0$ is (co)homologically like a one-complex, since by the Universal Coefficient Theorem we have, in such case, $H_2(K) = 0$ and $H^1(K) \approx H_1(K)$ torsion free. This fact aggravates the problem: since H^2 is not able to detect the existence of (non-collapsible) two-cells in K, is it able to detect the existence of strong surjections from K into a closed surfaces? Here, by a strong surjection we mean a map whose homotopy class contains only surjective maps.

Of course, the same question makes sense also in greater dimensions. In the 2000's, C. Aniz and D.L. Gonçalves approached the problem in dimension three. In [1], C. Aniz proved that every map from

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a 3-complex K with $H^3(K) = 0$ into $S^1 \times S^2$ is homotopic to a non-surjective map, but for Y the non-orientable S^1-bundle over S^2, there exists a strong surjection $f : K \to Y$ from such a 3-complex. In [2], C. Aniz proved that there is no strong surjection from a 3-complex K with $H^3(K) = 0$ into the orbit space of the 3-sphere S^3 with respect to the action of the quaternion group Q_8 determined by the inclusion $Q_8 \subset S^3$.

In 2008, D.L. Gonçalves introduced us to the problem, in dimension two, and conjectured that, as in dimension three, there should be a strong surjection from a two-complex with trivial top-cohomology onto some closed surface. It is noteworthy that the dimension two is often left out, in several contributions to topological root theory, since it does not permit the use of special techniques as obstruction theory. In order to attack our problem itself, also the Nielsen root theory is not feasible, despite the Wecken property for roots established by D.L. Gonçalves and P. Wong in [9], since there is no practical mechanisms for calculating the Nielsen root number.

Our approach is based in combinatorial group theory, in special equation in free groups, what has been used successfully in coincidence and root theory for maps between closed surfaces, as we see, for instance, in [3,7] and [8]. In [6], also via combinatorial group theory, we study indirectly the problem considering only maps into the real projective plane $\mathbb{R}\mathrm{P}^2$. We prove that for a finite and connected two-dimensional complex K, the condition $[K; \mathbb{R}\mathrm{P}^2]_* = 0$ implies $H^2(K) = 0$, and the opposite implication is true if the number of two-cells of K is equal to the first Betti number of its one-skeleton K^1.

In this article, we establish the conjecture proposed by D.L. Gonçalves. Specifically, we build a collection of two-complexes $K\langle p, q\rangle$ with trivial top-cohomology, for all coprime integers $p, q \geq 2$, and a strong surjection from each $K\langle p, q\rangle$ onto the torus $\mathbb{T} = S^1 \times S^1$.

To ensure the success of our approach, two things were particularly important: first, the necessary and sufficient condition, expressed in terms of equation in free groups, to a map from a two-complex into a closed surface to be strongly surjective, presented in [4]; second, the classical result of R.C. Lyndon and M.P. Schutzenberger, published in [10], which states that if a, b and c are elements of a free group and $a^m = b^n c^p$, for integers $m, n, p \geq 2$, then a, b and c are contained in a cyclic subgroup.

After we build the aforementioned collection of two-complexes with trivial top-cohomology and strong surjections from them onto the torus, we prove, in the final section of the article, that such collection is as simple as possible. We explain: each two-complex $K\langle p, q\rangle$ has five cells and the first Betti number of its one-skeleton is three; we prove that if K is a finite and connected two-complex with $H^2(K) = 0$ and either K has less than five cells or $\beta_1(K^1) \leq 2$, then there is no strong surjection from K into the torus.

2. On the equation $(Xw_1)^p(Yw_2)^q(Zw_3)^r = \mathbb{1}$

In this section, we prove a simple result in combinatorial group theory, which is directly used, in the next section, in the proof of our main theorem.

Let p, q and r be integers greater than or equal to two and let w_1, w_2 and w_3 be words in the free group $F^n = F(a_1, \ldots, a_n)$ of rank $n \geq 2$. Put $\mathbb{1}$ to be the empty word in F^n. In what follows we consider the equation

$$(Xw_1)^p(Yw_2)^q(Zw_3)^r = \mathbb{1} \tag{1}$$

in the free group F^n. A solution for Equation (1) is a group homomorphism $\phi : F(x_1, x_2, x_3) \to F^n$ for which the triad $(X, Y, Z) = (\phi(x_1), \phi(x_2), \phi(x_3))$ satisfies the equation. In such a case, we say also that such triad is a solution for Equation (1). Of course, Equation (1) has an obvious solution, namely, the triad $(w_1^{-1}, w_2^{-1}, w_3^{-1})$.

If ϕ is a solution for Equation (1) whose image is contained in a subgroup H of F^n, we say that ϕ is a solution for Equation (1) over H. We are particularly interested in the case in which H is the commutator subgroup of F^n. Therefore, let $\xi : F^n \to \mathbb{Z}^n$ be the abelianization homomorphism and, for each index