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# Right ideal decompositions of $G^*$

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ABSTRACT

Let $G$ be an infinite discrete group, let $\beta G$ be the Stone–Čech compactification of $G$, and let $G^* = \beta G \setminus G$. We show that if $G$ can be embedded algebraically in a compact zero dimensional second countable group (in particular, if $G = \mathbb{Z}$), then there are a decomposition $\mathcal{D}$ of $G^*$ into right ideals of $\beta G$ and a closed subsemigroup $T$ of $G^*$ containing all the idempotents such that $\mathcal{D}_T = \{R \cap T : R \in \mathcal{D}\}$ is a decomposition of $T$ into closed right ideals and $T/\mathcal{D}_T$ is homeomorphic to $\omega^*$.



## 1. Introduction

The operation of a discrete group $G$ extends to the Stone–Čech compactification $\beta G$ of $G$ so that the right translation $\beta G \ni x \mapsto xp \in \beta G$ is continuous for every $p \in \beta G$ and the left translation $\beta G \ni x \mapsto ax \in \beta G$ is continuous for every $a \in G$.

We take the points of $\beta G$ to be the ultrafilters on $G$, the principal ultrafilters being identified with the points of $G$, and $G^* = \beta G \setminus G$. The topology of $\beta G$ is generated by taking as a base the subsets of the form $\overline{A} = \{p \in \beta G : A \in p\}$, where $A \subseteq G$. For $p, q \in \beta G$, the ultrafilter $pq$ has a base consisting of subsets of the form $\bigcup\{xB_x : x \in A\}$, where $A \in p$ and $B_x \in q$.

Being a compact right topological semigroup, $\beta G$ has a completely simple kernel (= smallest two sided ideal). Completely simple implies that the smallest two sided ideal is a disjoint union of minimal right ideals and a disjoint union of minimal left ideals. The intersection of a minimal right ideal and a minimal left ideal

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is a group. In a semigroup with a completely simple kernel, every right (left) ideal contains a minimal right (left) ideal, and so it contains an idempotent.

The semigroup $\beta G$ is interesting both for its own sake and for its applications to Ramsey theory and to topological dynamics. An elementary introduction to $\beta G$ can be found in [4].

It has long been known that for every infinite group $G$, there is a decomposition (= partition) of $U(G)$, the subset of $\beta G$ consisting of uniform ultrafilters, into $2^{2^{|G|}}$ left ideals of $\beta G$ [1]. This result was strengthened in [6] by showing that there is a decomposition of $U(G)$ into $2^{2^{|G|}}$ closed left ideals of $\beta G$ (see also [3] and [7, Section 9.1]). In particular, for every countably infinite group $G$, there is a decomposition of $G^*$ into $2^{2^\omega}$ closed left ideals of $\beta G$.

In this paper we study decompositions of $G^*$ into right ideals of $\beta G$. In the case where $G$ is Abelian, every closed right ideal of $\beta G$ is a two sided ideal [4, Theorem 2.19], and so there is no nontrivial decomposition of $G^*$ into closed right ideals of $\beta G$.

In Section 2 of the paper we show that for every infinite group $G$ embeddable algebraically in a compact zero dimensional second countable group, there are a decomposition $\mathcal{D}$ of $G^*$ into right ideals of $\beta G$ and a closed subsemigroup $T$ of $G^*$ containing all the idempotents such that $\mathcal{D}_T = \{R \cap T : R \in \mathcal{D}\}$ is a decomposition of $T$ into closed right ideals and $T/\mathcal{D}_T$ is homeomorphic to $\omega^*$. By a compact group we always mean a compact topological group. A space is zero dimensional if it has a base of clopen sets. The group of integers $\mathbb{Z}$ can be embedded algebraically in a compact zero dimensional second countable group (for example, in the additive group of 2-adic integers), but the group of rationals $\mathbb{Q}$ cannot (see the paragraph before Question 2 in the end of the paper).

Our construction is based on some property of infinite compact zero dimensional second countable groups whose weak form, which we call the local cut point property, says that there are two open subsets $U, V \subseteq G \setminus \{1\}$ such that $U \cap V = \emptyset$, $1 \in (\operatorname{cl} U) \cap (\operatorname{cl} V)$, and $U \cup V \cup \{1\}$ is a neighborhood of 1. In Section 3 we show that a compact Abelian group has the local cut point property if and only if it is either infinite zero dimensional second countable or of the form $\mathbb{T} \times F$, where $\mathbb{T}$ is the circle group and $F$ is a finite group.

All spaces are assumed to be Hausdorff.

## 2. Construction

Let $S$ be a semigroup and let $T$ be a subsemigroup of $S$. A *right ideal decomposition* of $S$ is a decomposition of $S$ into right ideals of $S$. A right ideal $I$ of $S$ is an *extension* of a right ideal $R$ of $T$ if $I \cap T = R$. A right ideal decomposition $\mathcal{E}$ of $S$ is an *extension* of a right ideal decomposition $\mathcal{D}$ of $T$ if $\mathcal{D} = \{I \cap T : I \in \mathcal{E}\}$.

**Lemma 2.1.** *Let $S$ be a semigroup with a completely simple kernel, let $T$ be a subsemigroup of $S$ containing all the idempotents, and let $\mathcal{D}$ be a right (left) ideal decomposition of $T$. Then there is at most one extension of $\mathcal{D}$ to a right (left) ideal decomposition of $S$.*

**Proof.** Assume on the contrary that there are two distinct extensions $\mathcal{E}$ and $\mathcal{E}'$ of $\mathcal{D}$. Then there are distinct $I \in \mathcal{E}$ and $J \in \mathcal{E}'$ such that $I \cap T = J \cap T$. Without loss of generality one may suppose that $I \setminus J \neq \emptyset$. Pick $K \in \mathcal{E}'$ such that $(I \setminus J) \cap K \neq \emptyset$. Then $I \cap K$ is a right ideal of $S$ disjoint from $T$, so it contains an idempotent which is not in $T$, a contradiction. □

**Corollary 2.2.** *Let $T$ be a subsemigroup of $G^*$ containing all the idempotents and let $\mathcal{D}$ be a right (left) ideal decomposition of $T$. Then there is at most one extension of $\mathcal{D}$ to a right (left) ideal decomposition of $G^*$.*

Notice that if $\mathcal{D}$ is a decomposition of $G^*$ into right (left) ideals of $G^*$, then every member of $\mathcal{D}$ is also a right (left) ideal of $\beta G$. To see this, assume the contrary. Then there are distinct $I, J \in \mathcal{D}$, $p \in I$ and $x \in \beta G$ such that $px \in J$. But then $(px)p \in J$ and $p(xp) \in I$, since $xp \in G^*$, a contradiction.