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Strong Conley index over a phase space

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ABSTRACT

We extend the definition of the strong Conley index of a discrete dynamical system introduced in [1] to the strong Conley index over a phase space of a discrete dynamical system which strengthens the Conley index over a phase space defined in [2] and also generalizes the strong Conley index of a discrete dynamical system. We also prove the continuation property of the strong Conley index over a phase space.

1. Introduction

Charles Conley in his monograph "Isolated invariant sets and the Morse index" [3] introduced a (homotopy) index for a continuous dynamical system which he proved to be invariant under "continuation". This revolutionized the study of dynamical system in the sense that isolated invariant sets of "nearby" dynamical systems share similar nature. Ever since Conley's work, the index, now named as "the Conley index", has been generalized for discrete dynamical systems and has played a major role in the study of both continuous and discrete dynamical systems.

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In the above cited monograph Conley posed many open problems, one of them being the strengthening of the Conley index so that it can capture the essence of continuation more closely.

Motivated by this problem, we had defined a stronger version of the Conley index, namely, the strong Conley index of a discrete dynamical system, in [1], and proved that it was independent of the choice of filtration pairs for an isolated invariant set, and that it was continuation invariant.

We continue to further strengthen the Conley index of a discrete dynamical system in this paper to the notion of strong Conley index over a phase space of a discrete dynamical system, which, on the one hand, strengthens the Conley index over a phase space of a discrete dynamical system, defined in [2] and, on the other hand, strengthens the strong Conley index of a discrete dynamical system, defined in [1]. We also prove the Ważewski property and the continuation property of this new index.

The paper is arranged as follows: In section 2 we give references for the standard definitions on discrete dynamical systems including index (filtration) pairs and index (pointed space) maps and (strong) shift equivalence and (strong) Conley index etc.. In section 3 we recall (homotopy and homology) Szymczak category and introduce (strong) equivalence in the Szymczak category. In section 4 we introduce (homology) category of spaces over a base and (strong) (homology) M-equivalence. In section 5 we introduce index spaces and index maps over a phase space and prove the main technical Lemmas 5.1, 5.2, 5.5, 5.6, prove Theorems 5.4, 5.7, 5.8 and give Example 4.16. Finally in section 6 we define discrete (strong) Conley index over a phase space and prove its properties like Ważewski property Theorem 6.2 and the continuation property Theorem 6.3. We also give Example 4.33 contained in Remarks 4.32, 6.5, Lemma 6.8, Theorem 6.9 and Example 6.11, to show that strong Conley index over a phase space strengthens both the Conley index over a phase space and the strong Conley index. The concluding Example 6.12 compares the strong Conley index over a phase space and the strong homology Conley index over a phase space.

2. Preliminaries on discrete dynamical systems

Let X be a locally compact metric space and $f : X \to X$ a continuous map. The discrete dynamical system is a pair (X, f) and the dynamics is defined by the iterates of f. X is called the phase space. Throughout the paper phase spaces will be locally compact metric spaces.

We say that the dynamical system (X, f) has a full solution (or trajectory or orbit) through x if there exists a map $\sigma : \mathbb{Z} \to X$ such that $\sigma(0) = x$ and $\sigma(n + 1) = f(\sigma(n))$. If we write $x_n = \sigma(n), n \in \mathbb{Z}$, then a full trajectory through x can be rephrased as a sequence $\{x_n\}, n \in \mathbb{Z}$ such that $x_0 = x$ and $f(x_k) = x_{k+1}$ for all $k \in \mathbb{Z}$.

We recall some essential definitions from the papers [4,2] and the book [5].

Definition 2.1. ([2]) For an arbitrary subset $N \subset X$ we define

$$\operatorname{Inv}^n(N, f) := \{f^n(x) \in N \mid f^0(x), f^1(x), ..., f^{2n}(x) \in N\},$$
$$\operatorname{Inv}(N, f) := \{x \in N \mid \text{ a full solution through } x \text{ is contained in } N\}$$

which will be called the n-invariant and invariant part of N, respectively. $\operatorname{Inv}(N, f)$ is called the *maximal invariant subset* of N.

Now recall the definitions of *isolating neighborhood*, *isolated invariant set*, *isolating block*, *exit set* from [4,5]; *filtration pair*, *index pair*, *pointed space map*, *index map* from [4].

We will give here the definition of *elementary strong shift equivalence*, *strong shift equivalence* and *shift equivalence* over a semiring from [1], see also [6–9]. *Elementary strong shift equivalence*, and *strong shift equivalence* for a map from [1] and *shift equivalence* for a map from [7]; *h-elementary strong shift equivalence*, *h-strong shift equivalence* and *h-shift equivalence* for a map from [1].