Topology and its Applications 210 (2016) 263–268

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On minimal homeomorphisms on Peano continua

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ARTICLE INFO

Article history:
Received 5 June 2016
Received in revised form 20 July 2016
Accepted 23 July 2016
Available online 27 July 2016

Keywords:
Peano continuum
Minimal homeomorphism

ABSTRACT

Following the question of Artigue we construct a minimal homeomorphism $g\colon X \to X$ on a Peano continuum X with the following property: there exist a positive number ε and a dense G_δ subset E of X such that every non-trivial subcontinuum of X intersecting E expands under iterations of g to a continuum of diameter greater than ε.

1. Introduction

A continuum X is a nonempty, compact, connected metric space (X, dist). A continuum is non-trivial if it is not a singleton. Given a homeomorphism $f\colon X \to X$ we say that a non-trivial continuum $C \subset X$ is η-stable, $\eta > 0$, if $\mathrm{diam}(f^n C) \leq \eta$ for all $n \in \mathbb{Z}$.

We say that $f\colon X \to X$ is minimal if the set $\{f^n(x)\colon n \geq 0\}$ is dense in X for all $x \in X$. We denote by $\dim(X)$ the topological dimension of X (see [5]). In [2] Artigue has found possible non-trivial conditions under which the set of η-stable continua is nonempty for each $\eta > 0$.

Theorem 1.1. *([2]) If $f\colon X \to X$ is a minimal homeomorphism of a compact metric space $(X, dist)$ and $\dim(X) > 0$ then for all $\eta > 0$ there exists an η-stable continuum.*

A homeomorphism $f\colon X \to X$ is expansive if there is an $\eta > 0$ such that if $\mathrm{dist}(f^n(x), f^n(y)) < \eta$ for all $n \in \mathbb{Z}$ then $y = x$. In [7] Mañé proved that if a compact metric space X admits a minimal expansive homeomorphism then X is totally disconnected, i.e., every connected subset of X is a singleton.

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[1] The author was supported by the grant GAČR 14-06989P. The author is a junior researcher in the University Center for Mathematical Modeling, Applied Analysis and Computational Mathematics (Math MAC).

http://dx.doi.org/10.1016/j.topol.2016.07.022
0166-8641/

In [6] Kato introduced a generalization of expansivity to continuum-wise expansivity that allowed him to extend several results of expansive homeomorphisms, including those obtained in [7].

Since it is known that $\dim(X) > 0$ if and only if X has a non-trivial connected component [5], Theorem 1.1 provides an alternative proof of Mañé's result.

The set of all η-stable continua will be denoted by $C_\eta(X)$. We say that a point $x \in X$ *belongs to stable continua* if for each $\eta > 0$ there is a continuum $C \in C_\eta(X)$ containing x.

In [4] Floyd gave an example of a compact subset $X \subset \mathbb{R}^2$ and a minimal homeomorphism $f\colon\ X \to X$. Each connected component of X is an interval, some of them trivial, i.e., singletons. Therefore, X is a nonhomogeneous space of positive topological dimension. This example shows that in a minimal dynamical system there can be points not belonging to a non-trivial η-stable continuum for any $\eta > 0$ (the trivial components) – compare this fact with Theorem 1.1.

Considering the conclusion contained in Theorem 1.1, Artigue in [2, Remark 2.6] has asked, assuming that X is a Peano continuum and $f\colon\ X \to X$ a minimal homeomorphism: Is it true that every point from X belongs to stable continua? The purpose of this note is to show that the answer to the above question is negative. We also concern the Borel type of the set of all points belonging to stable continua and evaluate its Baire category.

2. Points belonging to stable continua

As general reference for all used notions from continuum theory – see [8].

We only recall that for a continuum X, we consider the following hyperspaces equipped with the Hausdorff metric and the corresponding topology:

$$C(X) = \{A \subseteq X\colon\ A \text{ is a continuum}\},$$

$$C(X,n) = \{C \in C(X)\colon\ \operatorname{diam}(C) \geq 1/n\},\ n \in \mathbb{N},$$

$$C(x) = \{C \in C(X)\colon\ x \in C\},\ x \in X.$$

We start with detection of the Borel class of the set of all points belonging to stable continua and, for f minimal, also its topological size.

Proposition 2.1. *For a continuum X, let $f\colon\ X \to X$ be a homeomorphism. The set S of all points belonging to stable continua is of type $F_{\sigma\delta}$. If f is minimal and $S \subsetneq X$ then S is a meager set.*

Proof. For $\eta > 0$ let us put

$$I_{\eta,n} = \{x \in X\colon\ C(x) \cap C_\eta(X) \cap C(X,n) \neq \emptyset\}. \tag{1}$$

We show that the set $I_{\eta,n}$ is closed. If $(x_i)_i$ is a sequence of elements from $I_{\eta,n}$ that converges to a point $x \in X$, then using (1) we can consider a sequence $(C_i)_i$ with $C_i \in C(x_i) \cap C_\eta(X) \cap C(X,n)$ for each i. Without loss of generality we can assume that $(C_i)_i$ converges in the hyperspace $C(X)$ to a continuum C. Then we immediately obtain that $C \in C(x) \cap C(X,n)$. Since the map f is a homeomorphism, from definition of the Hausdorff metric on $C(X)$ we obtain also $\operatorname{diam}(f^nC) \leq \eta$ for all $n \in \mathbb{Z}$, i.e., $C \in C_\eta(X)$. It implies that $C \in C(x) \cap C_\eta(X) \cap C(X,n)$ hence $x \in I_{\eta,n}$.

If we put

$$I_\eta = \bigcup_{n \geq 1} I_{\eta,n},$$