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Quantum knot mosaics and the growth constant ☆

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A B S T R A C T

Lomonaco and Kauffman introduced a knot mosaic system to give a precise and workable definition of a quantum knot system, the states of which are called quantum knots. This paper is inspired by an open question about the knot mosaic enumeration suggested by them. A knot n-mosaic is an $n \times n$ array of 11 mosaic tiles representing a knot or a link diagram by adjoining properly that is called suitably connected. The total number of knot n-mosaics is denoted by D_n which is known to grow in a quadratic exponential rate. In this paper, we show the existence of the knot mosaic constant $\delta = \lim_{n\to\infty} D_n^{\frac{1}{n^2}}$ and prove that

$$4 \le \delta \le \frac{5+\sqrt{13}}{2} \ (\approx 4.303).$$

1. Preliminaries

The quantum knot system was developed by Lomonaco and Kauffman to explain how to make quantum information versions of mathematical structures in [4,5]. They build a knot mosaic system to set the foundation for a quantum knot system, based on the planar projections of knots and the Reidemeister moves.

Throughout this paper the term ‘knot’ means either a knot or a link. An example of a knot mosaic is shown in Fig. 1 (a). Knot mosaics are constructed by using 11 mosaic tiles, listed in Fig. 1 (b).

This paper is inspired by an open question (9) about the knot mosaic enumeration proposed in [5]. The enumeration of knot mosaic is not only an interesting problem in its own right but is also of considerable importance in the quantum knot theory. Let D_n denote the total number of knot n-mosaics. The author, Hong, Lee and Lee announced several results on D_n in the series of papers [1–3,7]. Based upon the results,

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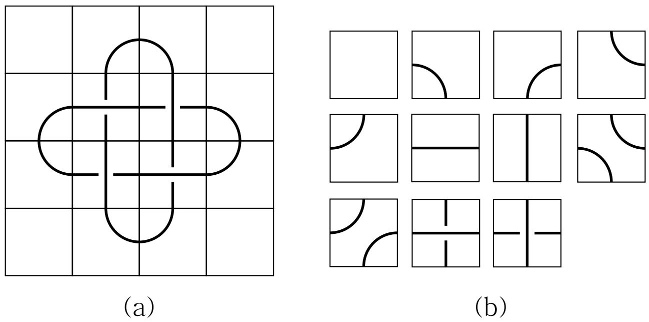

Fig. 1. An example of a knot mosaic and 11 mosaic tiles.

D_n is known to grow in a quadratic exponential rate. We consider the behavior of the growth rate. The limit, if it exists,

$$\delta = \lim_{n\to\infty} D_n^{\frac{1}{n^2}}$$

is called the *knot mosaic constant*.

Theorem 1. *The knot mosaic constant δ exists. Furthermore,*

$$4 \leq \delta \leq \frac{5+\sqrt{13}}{2} \ (\approx 4.303).$$

As a previous result, lower and upper bounds on D_n for $n \geq 3$ were established as follows in [1]:

$$\frac{2}{275}(9 \cdot 6^{n-2}+1)^2 \cdot 2^{(n-3)^2} \ \leq \ D_n \ \leq \ \frac{2}{275}(9 \cdot 6^{n-2}+1)^2 \cdot (4.4)^{(n-3)^2} \qquad (*)$$

These bounds suggested that δ lies between 2 and 4.4.

This paper is organized as follows. In Section 2, we give precise definition of knot mosaics with a slight generalization and previously known results about the enumeration of knot mosaics. In Section 3, the existence of the knot mosaic constant δ is provided by applying an extended version of Fekete's Lemma. In Section 4, we rigorously find lower and upper bounds of δ with heavy reliance on the main theorem of [7].

2. Enumeration of knot mosaics

We begin by presenting the basic notion of knot mosaics and then introduce previously known results about the enumeration of knot mosaics.

Definition 1. For positive integers m and n, an (m, n)-*mosaic* is an $m \times n$ array $M = (M_{ij})$ of 11 mosaic tiles depicted in Fig. 1 (b).

This definition is a rectangular version of the definition of an n-*mosaic* that is an $n \times n$ array of mosaic tiles. Obviously the set of all (m, n)-mosaics has 11^{mn} elements.

A *connection point* of a mosaic tile is defined as the midpoint of a mosaic tile edge that is also the endpoint of a curve drawn on the tile. Then the first mosaic tile has zero, the next six tiles with exactly one curve inside have two, and the last four tiles have four connect points. A mosaic is called *suitably connected* if any pair of mosaic tiles lying immediately next to each other in either the same row or the same column have or do not have connection points simultaneously on their common edge.