Topology and its Applications 201 (2016) 13–17

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On the properties of subsets of Tychonoff products ☆

A.A. Gryzlov

Department of Algebra & Topology, Udmurt State University, Universitetskaya st., 1, Izhevsk, Udmurtiya 426034, Russia

A R T I C L E I N F O

Article history:
Received 31 December 2014
Accepted 19 April 2015
Available online 21 December 2015

MSC:
54A25
54B10

Keywords:
Tychonoff product
Cardinality
Projection

A B S T R A C T

We consider properties of subsets of Tychonoff products and their relations with properties of projections. By this way we get some cardinal inequalities for topological spaces.

1. Introduction

The cardinality of a topological space is one of the main characteristic of the space.

Archangel'skiĭ's theorem [2] states that $|X| \leq 2^{\chi(X)}$ for a compact (and even a finally compact) T_2-spaces. Moreover, $|X| \leq 2^{\psi(X)}$ for a compact T_1-space [4].

In [5] we considered properties of subsets of Tychonoff products and their projections.

Using these properties, we proved some cardinality inequalities.

Here we prove Theorem 3.1 from which it directly follows, in particular, Archangelskiĭ's inequality for compact T_2-spaces and also the inequality for locally compact connected T_2-spaces.

2. Preliminaries

Notation used in the paper is standard. By $|X|$, $\chi(X)$, $\psi(X)$, $w(X)$ we denote cardinality, character, pseudocharacter and weight of a space X respectively. By $[B]$ we denote the closure of a set B. We assume that all spaces are T_2-spaces.

☆ Supported by Ministry of Education and Science of the Russian Federation (project number 2003).
E-mail address: gryzlov@udsu.ru.

http://dx.doi.org/10.1016/j.topol.2015.12.023
0166-8641/

All definitions and notions used in the paper can be found in [1,3].

Recall some notions and facts (see [5]).

Let $\prod_{\alpha\in A} Y_\alpha$ be the Tychonoff product of topological spaces Y_α $(\alpha \in A)$.

For a subset $A' \subseteq A$ the space $\prod_{\alpha\in A'} Y_\alpha$ is called an A'-face of $\prod_{\alpha\in A} Y_\alpha$.

A mapping $\pi_{A'}: \prod_{\alpha\in A} Y_\alpha \to \prod_{\alpha\in A'} Y_\alpha$, defined by the rule:

> if $x = \{x_\alpha : \alpha \in A\} \in \prod_{\alpha\in A} Y_\alpha$, then $\pi_{A'}(x) = \{x_\alpha : \alpha \in A'\} \in \prod_{\alpha\in A'} Y_\alpha$, is called an A'-projection of $\prod_{\alpha\in A} Y_\alpha$.

If $X \subseteq \prod_{\alpha\in A} Y_\alpha$ and $A' \subseteq A$, then the restriction $p_{A'} = \pi_{A'}|_X$ is called a A'-projection of X, $p_{A'}: X \to p_{A'}(X) \subseteq \prod_{\alpha\in A'} Y_\alpha$.

Let $X \subseteq \prod_{\alpha\in A} Y_\alpha$.

For every point $x \in X$ fix the set

$$A(x) \subseteq A,\ |A(x)| \le \chi(X)$$

and the family $B(A(x))$, consisting of sets W of the type

$$W = U_{\alpha_1} \times .. \times U_{\alpha_n} \times \prod_{\alpha\in A\setminus\{\alpha_1,..,\alpha_n\}} Y_\alpha,$$

where $\alpha_1, .., \alpha_n \in A(x), U_{\alpha_i} \subseteq Y_{\alpha_i}$ are open for $i = 1, \dots, n, n \in \omega$, such that the family

$$\{W \cap X : W \in B(A(x))\}$$

is the base of x in the space X.

Lemma 2.1. *([5]) Let $X \subseteq \prod_{\alpha\in A} Y_\alpha$ and $x \in X$. Then there is $A' \subseteq A$, $|A'| \le \psi(x, X)$ such that $|p_{A'}^{-1}(p_{A'}(x))| = 1$.*

From Lemma 2.1 it follows that for every point $x \in X$, $X \subseteq \prod_{\alpha\in A} Y_\alpha$, we have $|p_{A(x)}^{-1}(p_{A(x)}(x))| = 1$, for the $A(x)$-projection $p_{A(x)}: X \to p_{A(x)}(X)$.

Definition 2.1. A continuous mapping $f: X \to Y$ is called a *locally quotient* (l-quotient) if the following is true:

> if a closed set $A \subseteq X$ is such that $A = f^{-1}(f(A))$, then the for every $x \in A$ there is a neighbourhood Ox such the $f([Ox \cap A])$ is closed in Y.

A quotient mapping $f: X \to Y$, and therefore a closed or an open mapping, is l-quotient. Then if $f: X \to Y$ is a continuous mapping, X, Y are T_2-spaces and X is a compact or locally compact space, then the mapping f is l-quotient.