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A categorical approach to convergence: Compactness

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A R T I C L E   I N F O

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A convergence class is introduced on every object of a given category by using certain generalized nets for expressing the convergence. The obtained concrete category is then investigated whose objects are the pairs consisting of objects of the original category and convergence classes on them and whose morphisms are the morphisms of the original category that preserve the convergence. We define, in a natural way, separation and compactness of objects of the concrete category under investigation and study their behavior.

1. Introduction

The study of topological structures on (objects of) categories represents an important branch of categorical topology. It was initiated by D. Dikranjan and E. Giuli [4] who introduced and studied closure operators on categories. These operators were then investigated by a number of authors (see [6] and the references therein) who contributed to the development of the theory of categorical closure operators. In particular, some of these authors studied separation and compactness with respect to a categorical closure operator – see e.g. [2,3,5]. Later on, some more topological structures on categories were introduced and studied including convergence structures [16], neighborhood structures [10], and interior operators [20].

In the classical approach to the study of topological structures on (objects of) categories, categories with a given topological structure are considered and investigated. This approach is used also in [16] for the study of convergence on categories and related separation and compactness. Quite a different approach was used in [13–15] where concrete categories over *Set* were studied obtained by providing every set with a convergence and introducing continuous, i.e., convergence preserving maps. In the present paper, we use the approach of [13–15] but, instead of Set, an arbitrary category is considered (with no topological structure

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in general) and a convergence is newly defined for each of its objects. We obtain a concrete category over the original one whose objects are pairs consisting of objects of the original category and convergence on them and whose morphisms are the morphisms of the original category that preserve the convergence. Basic properties of the concrete category are investigated in [17]. In the present paper, we focus on the study of naturally defined convergence separation and compactness of objects of the concrete category obtained. We will show that the separation and compactness behave analogously to and even better than the usual topological separation and compactness.

2. Preliminaries

For the convenience of the reader, we repeat all the relevant definitions from [17], which makes our paper self-contained. For the categorical terminology used see [1] and [12].

Throughout the paper, we consider a category $\mathcal{K}$ with a terminal object $1_{\mathcal{K}}$. Further, we assume there is given a non-empty category $\mathcal{S}$ and a functor $\mathcal{F} : \mathcal{S} \to \mathcal{K}$.

Definition 1. For an arbitrary $\mathcal{K}$-object K, by an $\mathcal{F}$-*net* in K we understand any object of the comma category $(\mathcal{F} \downarrow K)$, i.e., any object $\mathcal{F}$-over K. Given a pair $\langle S, f\rangle, \langle T, g\rangle$ of $\mathcal{F}$-nets in K, $\langle S, f\rangle$ is said to be a *subnet* of $\langle T, g\rangle$ provided that there is a morphism from $\langle S, f\rangle$ to $\langle T, g\rangle$ in $(\mathcal{F} \downarrow K)$.

Example 1. Here and also in the other examples, we will denote by *Dir* the category of directed sets and cofinal maps and by $\underline{\omega}$ the subcategory of *Dir* whose only object is the least infinite ordinal ω and whose morphisms are the isotone injections.

(1) Let $\mathcal{K} = \mathit{Set}$, $\mathcal{S} = \mathit{Dir}$ and let $\mathcal{F} : \mathit{Dir} \to \mathit{Set}$ be the forgetful functor. Then $\mathcal{F}$-nets in a $\mathcal{K}$-object X and their subnets are the usual nets in the set X and their subnets (see e.g. [11]). If we replace *Dir* by $\underline{\omega}$, then we get the usual sequences and subsequences.

(2) Let $\mathcal{S}$ be a construct and let $\mathcal{F} : \mathcal{S} \to \mathit{Set}$ be the forgetful functor. Then $\mathcal{F}$-nets in a $\mathcal{K}$-object X and their subnets coincide with $\mathcal{S}$-nets in X and their subnets discussed in [13–15].

(3) The $\mathcal{B}$-nets in a set X and their $\mathcal{B}$-subnets from [21] are nothing but $\mathcal{F}$-nets in X and their subnets where $\mathcal{B}$ is a subconstruct of *Dir* and $\mathcal{F} : \mathcal{B} \to \mathit{Set}$ is the forgetful functor.

(4) Let $\mathcal{K} = \mathit{Set}$, let $\mathcal{S}$ be the category of compact Hausdorff topological spaces (and continuous maps) and let $\mathcal{F} : \mathcal{S} \to \mathit{Set}$ be the forgetful functor. A quasi-topology [18] on a set X is nothing but a collection $(Q(S, X))_{S \in \mathcal{S}}$ where, for each $\mathcal{S}$-object S, $Q(S, X)$ is a set of $\mathcal{F}$-nets $\langle S, f\rangle$ in X satisfying certain given axioms.

For any $\mathcal{K}$-object K, we denote by $\cong$ the usual equivalence between subobjects of K (i.e., monomorphisms in $\mathcal{K}$ with the codomain K) and by K^* the class of all ($\cong$-equivalence classes of) points of K, i.e., subobjects of K whose domains are terminal objects.

Definition 2. Let K be a $\mathcal{K}$-object. A subclass $\pi \subseteq Ob(\mathcal{F} \downarrow K) \times K^*$ is said to be a *convergence class* on K if the following two conditions are satisfied:

(i) If $\langle S, f\rangle$ is an $\mathcal{F}$-net in K such that f factors through a point $x \in K^*$ (i.e., f is a constant morphism), then $(\langle S, f\rangle, x) \in \pi$.
(ii) If $(\langle S, f\rangle, x) \in \pi$, then $(\langle T, g\rangle, x) \in \pi$ for every subnet $\langle T, g\rangle$ of $\langle S, f\rangle$.

If π is a convergence class on a $\mathcal{K}$-object K, then we write $\langle S, f\rangle \overset{\pi}{\to} x$ instead of $(\langle S, f\rangle, x) \in \pi$ and say that $\langle S, f\rangle$ *converges* to x with respect to π. An $\mathcal{F}$-net $\langle S, f\rangle$ in K is said to be *convergent* w.r.t. π if there is a point $x \in K^*$ with $\langle S, f\rangle \overset{\pi}{\to} x$.