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On asymptotic power dimension

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ABSTRACT

We consider the asymptotic power dimension, i.e., the asymptotic dimension with control power function. It is proved that this dimension is invariant under the coarse bi-Hölder transformations.

We also establish estimates for the power asymptotic dimension of the symmetric and hypersymmetric products.

1. Introduction

The asymptotic Assouad–Nagata dimension of a metric space was introduced in [3]. By the definition, the asymptotic Assouad–Nagata dimension of a metric space (X, ρ) does not exceed n ($\mathrm{asdim}_{AN} X \le n$) if there exist $c > 0$ and $r_0 > 0$ such that for all $R > r_0$ one can find R-disjoint and cR-bounded families $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_n$ of subsets of X such that $\bigcup_{i=0}^{n} \mathcal{U}_i$ covers X (see the details below). The asymptotic Assouad–Nagata dimension is investigated by many authors (see, e.g., [1,2,8]).

In this paper we consider an analogous concept to this dimension. Namely, in our case the control function is the power function $x \mapsto x^{\alpha}$, where $\alpha > 0$. One of the motivations of this notion, in addition to its naturalness, consists in its relations to the important class of Hölder maps. The obtained dimension is in

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between the asymptotic dimension and the asymptotic Assouad–Nagata dimension. An example is provided of a space with these three dimensions different.

We also establish estimates for the power asymptotic dimension of the symmetric and hypersymmetric products.

2. Dimensions with power functions controlling diameter

A family $\mathcal{A}$ in a metric space (X, d) is said to be D-disjoint, where $D > 0$, if $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\} \geq D$, for every distinct $A, B \in \mathcal{A}$. A family $\mathcal{A}$ is called B-bounded, for $B > 0$, if

$$\mathrm{mesh}(\mathcal{A}) = \sup\{\mathrm{diam}(A) \mid A \in \mathcal{A}\} \leq B.$$

By $B_r(x)$ we denote the ball of radius r centered at x.

Let (X, ρ) be a metric space and let n be a nonnegative integer.

We use the notation (λ, B)-dim$X \leq n$ in the meaning that there exist λ-disjoint and B-bounded families $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$ of subsets of X such that $\bigcup_{i=0}^{n} \mathcal{U}_i$ is a cover of X. If for every $\lambda > 0$ there exists $B(\lambda) > 0$ such that (λ, B)-dim$X \leq n$, we say that $\lambda \mapsto B(\lambda)$ is an n-dimensional control function for X.

Definition 2.1. We say that the power dimension of X does not exceed n ($\mathrm{asdim}_P X \leq n$) if there exist $\alpha > 0$ and $r_0 > 0$ such that (r, r^α)-dimX $\leq n$ for all $r > r_0$.

As usual we say that $\mathrm{asdim}_P X = n$ if $\mathrm{asdim}_P X \leq n$ and it is not true that $\mathrm{asdim}_P X \leq n - 1$.

Theorem 2.2. *For any metric space X the following inequality holds:*

$$\mathrm{asdim}_P X \leq \mathrm{asdim}_{AN} X.$$

Proof. Let $\mathrm{asdim}_{AN} \leq n$ and c and r_0 fulfill the definition. Let $\alpha = 2$ and $\lambda_0 = \max(r_0, c)$. Then for $R > \lambda_0$ we have (R, cR)-dim$X \leq n$ and so (R, R^α)-dim$X \leq n$ as $R^2 > cR$. That means that $\mathrm{asdim}_P X \leq n$, so $\mathrm{asdim}_P X \leq \mathrm{asdim}_{AN} X$. □

It is well-known that, for any metric d on a set X, the function $d' = \ln(1 + d)$ is also a metric on X. The following result is, in some sense, a special case of [8].

Theorem 2.3. *Let (X, d) be a metric space. Then* $\mathrm{asdim}_P(X, d) = \mathrm{asdim}_{AN}(X, d')$.

Proof. Suppose that $\mathrm{asdim}_P(X, d) \leq n$, then there exist $R_0 > 0$ and $\alpha > 0$ such that (R, R^α)-dim$(X, d) \leq n$. Without loss of generality, one may assume that $\alpha \geq 1$.

Let $r_0 = \ln(1 + R_0)$. Using the inequality $1 + R^\alpha \leq (1 + R)^\alpha$, one easily concludes that (X, d') is $(r, \alpha r)$-dim$(X, d') \leq n$. Therefore, $\mathrm{asdim}_P(X, d) \geq \mathrm{asdim}_{AN}(X, d')$.

The reverse inequality can be proved by similar arguments. □

Remark 2.4. Note that the obtained metric space (X, d') is hyperbolic in the sense of Gromov [4].

Remark 2.5. Theorem 2.3 does not mean that one can immediately obtain the properties of the asymptotic power dimension out of those of the asymptotic Assouad–Nagata dimension. The reason is that, in general, there is no way from the latter to the former.

Given $A \subset X$, we denote by $B_r(A)$ the r-neighborhood of A, i.e., $B_r(A) = \cup\{B_r(x) \mid x \in A\}$.