Topology and its Applications 201 (2016) 269–273

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Preservation of complete metrizability by covering maps

Alexey Ostrovsky

Saint-Petersburg State University of Aerospace Instrumentation, Russian Federation

ARTICLE INFO

Article history:
Received 31 December 2014
Accepted 16 April 2015
Available online 4 January 2016

MSC:
primary 54E40, 54C05, 54C10
secondary 26A15, 46A30, 97I30

Keywords:
Compact-covering
Polish space
Complete metric
n-to-one maps

ABSTRACT

A map $f : X \to Y$ is said to be w-covering if for every ordinal α and every compact $S \subset Y$ such that the α-th Cantor derivative $(S)^\alpha$ of S is a singleton $\{y\}$ there is a compact subset E of X such that $f(E) \subset S$ and $(f(E))^\alpha = \{y\}$.
We investigate the preservation of completely metrizability by various kinds of w-covering maps.

We assume that all spaces are separable and metrizable and all maps $f : X \to Y$ are continuous onto $Y = f(X)$.

Recall that a map $f : X \to Y$ is **compact-covering** [1] (respectively, **s-covering** [10] or countable-compact-covering) if for every compact (respectively, countable compact) set $F \subset Y$, there is a compact set $E \subset X$ such that $f(E) \supset F$.[1]

In 1968 Michael and Stone proved [6]:

- compact-covering images of irrationals are F_{II}-spaces.[2]

In 1971 the author has proved a similar assertion involving s-covering maps [10]:

- s-covering images of F_{II}-spaces are F_{II}-spaces. (1)

E-mail address: Alexei.Ostrovski@gmx.de.

[1] Since f is continuous, and thus $f^{-1}(F)$ is closed, $f(E) \supset F$ is equivalent to $f(E) = F$.

[2] According to Hausdorff, a space X is a F_{II}-spaces if each of its subspaces Z of the first category (on Z) is not closed in X.

http://dx.doi.org/10.1016/j.topol.2015.04.019

Several authors [3,19,11] have shown independently in the early 70's that the image of a complete metric space under such a map is itself completely metrizable (see also [2,4,5,8,20,18,17]).

The starting idea of the present research was the observation that the proof of the assertion (1) remains valid if we replace "s-covering" with "w-covering".

Since F_{II}-spaces are very close to Polish spaces, it would be natural to ask:

$$\text{Does every } w\text{-covering map preserve complete metrizability?} \tag{2}$$

Our goal is to show that this relates to the preservation of complete metrizability by different kinds of covering maps (see References) and their derivatives.

Before defining w-covering maps, we introduce the following notion:

A countable compact subset of Y is said to be **compact of order** α **with top** y, and is denoted by $S_\alpha(y)$, if its α-th derived set $(S_\alpha(y))^\alpha$ is the singleton $\{y\}$.[3]

In particular, this means that $S_0(y) = \{y\}$. Obviously, in case of separable metric spaces $\alpha < \aleph_1$.

Definition 1. A map $f : X \to Y$ is said to be w**-covering** if for every $S_\alpha(y) \subset Y$ there is a compact set $E \subset X$ such that $f(E) \subset S_\alpha(y)$ is a compact set of order α with top y.

The main difference between s-covering and w-covering is that in the latter no surjectivity is asked.

A map $f : X \to Y$ is said to be h**-covering** if for every ordinal α and every $S_\alpha(y) \subset Y$ there is a compact set $E \subset X$ such that $f|E$ is one-to-one and $f(E) \subset S_\alpha(y)$ is a compact set of order α with top y.

Obviously, we have:

- f is compact-covering $\Longrightarrow f$ is s-covering $\Longrightarrow f$ is w-covering;
- f is h-covering $\Longrightarrow f$ is w-covering.

An example given by the author in [12] shows:

- f is w-covering $\not\Longrightarrow f$ is s-covering.

In this example the map $f : X \to S_1(y)$ is a composition of an open and a perfect map. Since open and closed maps are stable and every composition of stable maps is stable [15] it follows that a composition of s-covering maps can be non s-covering.

Recall that a map is **stable** [15,7] if one can assign to every point $y \in Y$ a system η_y of subsets open in X, intersecting $f^{-1}(y)$ and such that

- $X \in \eta_y$ and
- for every $U \in \eta_y$ and open $V \supset f^{-1}(y) \cap U$ there is a deleted neighborhood $\dot{O}(y)$ of y such that $V \in \eta_{y'}$ for every point $y' \in \dot{O}(y)$.

The following two propositions have been established in [15, Proposition E] and [15, Theorem 0]:

Proposition 1. *Every s-covering map is stable.*

Proposition 2. *Stable maps $f : X \to Y$ preserve complete metrizability.*

[3] The derived set $(X)'$ is the set of all limit points of X. Recall that $(X)^0 = X$, $(X)^{\alpha+1} = (X^\alpha)'$ and $(X)^\alpha = \bigcap_{\beta<\alpha} X^\beta$ for limit ordinals α.