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On product of p-sequential spaces

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The product of finitely many regular p-compact p-sequential spaces is p-compact p-sequential for any free ultrafilter p as it follows from [5]. In the paper is produced an example of a Hausdorff p-compact p-sequential space whose square is not p-sequential. It is also given an example of a space which is sP-radial, wP-radial, vwP-radial for any $P \subset \mu(\tau)$ but its square is neither sP-radial nor wP-radial nor vwP-radial space.

All spaces in this paper are assumed to be Hausdorff. Any infinite cardinal is associated with the initial ordinal of the same power. Let τ be any infinite cardinal. The Stone–Čech compactification of the discrete space τ is denoted as $\beta\tau$ and its remainder $\beta\tau \setminus \tau$ is identified with the set of all free ultrafilters on τ and $\mu(\tau) = \{p \in \beta\tau \setminus \tau : |A| = \tau \text{ for each } A \in p\}$ will denote the set of all uniform ultrafilters on τ [3].

Bernstein in [2] introduced the notions of a p-limit point and a p-compact space for any free ultrafilter p on ω, i.e. on the discrete space of positive integers. Kombarov [6] introduced the notions of P-compactness and P-sequentiality, where $P \subset \beta\omega \setminus \omega$ is a nonempty set of free ultrafilters on ω and he proves in [7] that the countable product of regular p-compact p-sequential spaces is a p-compact p-sequential one. Saks [4] generalizes the notion of p-limit to nets as follows: if $p \in \beta\tau \setminus \tau$ and $(x_\alpha : \alpha < \tau)$ is a τ-sequence in X, then a point x is a p-limit point of $(x_\alpha : \alpha < \tau)$, if for every neighborhood O of x $\{\alpha : x_\alpha \in O\} \in p$, denoted as $x = p\text{-}lim\ x_\alpha$ and he defines there a p-compact space as a space in which every τ-sequence has a p-limit point (or shortly: τ-sequence p-converges).

Following Kombarov [6] we call a space (X, σ) to be p-sequential or in other terminology known as a p-pseudo-radial space (see [5]), if for any nonclosed $A \subset X$ there is a point $x \notin A$ which is a p-limit point for some τ-sequence $(x_\alpha : \alpha < \tau) \subset A$.

Kočinac proves in [5] the theorem which implies that the product of finitely many regular p-compact p-sequential spaces is again a p-compact p-sequential space. We construct here two Hausdorff p-compact

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p-sequential spaces whose product is not p-sequential which will mean that the requirement of regularity in the mentioned theorems of Kombarov and Kočinac is essential and this construction allows to create a p-compact p-sequential space whose square is not p-sequential. To produce these spaces we will use terminology and constructions from [1].

Let $\{X_\alpha : \alpha < \tau\}$ be a family of pairwise disjoint sets each of power τ. Assume $\{y_\alpha : \alpha < \tau\}$ is a family of pairwise different objects with none of them in $T = \cup\{X_\alpha : \alpha < \tau\}$. We put $Y_\alpha = X_\alpha \cup \{y_\alpha\}$, $Y = \cup\{Y_\alpha : \alpha < \tau\}$ and $Q = \{y_\alpha : \alpha < \tau\}$. Let z^* be some object not in Y. We set $Z = T \cup \{z^*\}$ and define a mapping φ from Y onto Z by the following rule: $\varphi(y) = y$ if $y \in T$ and $\varphi(y) = z^*$ if $y \in Q$. We endow each Y_α with the following topology σ_α: if $A \subset X_\alpha$ then $A \in \sigma_\alpha$ and if M is any subset of X_α of power less than τ then $Y_\alpha \setminus M \in \sigma_\alpha$. So $(Y_\alpha, \sigma_\alpha)$ turns to be a topological space with the only nonisolated point y_α. W.l.o.g. one can consider $p \in \mu(\tau)$. Thus, the topological space $(Y_\alpha, \sigma_\alpha)$ is p-sequential for any $p \in \mu(\tau)$. Topology σ on Y is defined as a free union of a family of topologies $\{\sigma_\alpha : \alpha < \tau\}$, so the space (Y, σ) becomes p-sequential. We equip the set Z with the factor topology with respect to the mapping φ. So the space Z is a p-sequential space as a factor image of a p-sequential space [5] and $|Z| = \tau$. The space Z is an analogue of the Frécher–Urysohn fan, extended to any cardinal.

Let S be a set consisting of all possible τ-sequences $(x_\alpha : \alpha < \tau)$ in Z such that $x_\alpha \in \varphi(X_\alpha)$ for each $\alpha < \tau$. It is easily seen that $|S| = 2^\tau$. For any τ-sequence $M = (x_\alpha : \alpha < \tau)$, $M \in S$ let $F_M = \{\{x_\alpha : \alpha > \beta\}, \beta < \tau\}$. It is clear that F_M is a centered system of subsets of power τ and the family $\{F_M : M \in S\}$ forms a τ-singular system at z^* (see [1]).

Let a_M be a new object not in M. We set $M' = M \cup \{a_M\}$ and equip M' with a topology where each subset of M is open and the family $\{\{a_M \cup \{x_\alpha : \alpha > \beta\}\}, \beta < \tau\}$ is declared to be a base of open neighborhoods of the point a_M. Obviously that $|\{x_\alpha : \alpha < \beta\}| < \tau$ for any $\beta < \tau$ so M' becomes p-sequential for any $p \in \mu(\tau)$ with one nonisolated point a_M. Putting $W = \cup\{M' : M \in S\}$ endowed with the topology of a free union one can transform the topological space W into some space V with one nonisolated point t^* by repeating the same steps which were used to transform the space Y into the space Z. In this way we get two spaces Z and V each being p-sequential. Now using Theorem 3.5 in [1] we obtain the following inequality: $t((z^*, t^*), Z \times V) > \tau$ and taking into account that the tightness of a p-sequential space does not exceed τ [5] one can say that the space $Z \times V$ is not p-sequential. These two spaces Z and V are not p-compact so to get one of the required examples it is enough to create two Hausdorff p-compact p-sequential extensions of Z and V. Let (X, σ) be any topological space.

Definition 1. A subset $O \subset X$ is called p-sequentially open if $x \in O$ and $x = p\text{-}lim\ x_\alpha$ for some τ-sequence $(x_\alpha : \alpha < \tau)$ imply that $\{\alpha : x_\alpha \in O\} \in p$.

It is clear that the intersection of a finite number of p-sequentially open sets is p-sequentially open and the union of any number of p-sequentially open sets is again p-sequentially open so we can say that the set of all p-sequentially open sets in (X, σ) forms some topology which will be denoted by symbol σ_p. Obviously that each open set is p-sequentially open so we get the following statement.

Proposition 1. *(X, σ_p) is a topological space and $\sigma \subset \sigma_p$.*

For every subset $A \subset X$ we define the following set $p(A) = A \cup \{x \in X$: there is some τ-sequence $(x_\alpha : \alpha < \tau) \subset A$ such that $x = p\text{-}lim\ x_\alpha\}$.

Definition 2. A subset $A \subset X$ is said to be p-sequentially closed iff $A = p(A)$.

Proposition 2. *A subset A in a topological space (X, σ) is p-sequentially closed iff $X \setminus A$ is p-sequentially open.*