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On some generalizations of metric, normed, and unitary spaces

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ABSTRACT

Earlier the author offered new notions of multimetric, multinormed and multiunitary spaces which are generalizations of metric, normed and unitary spaces respectively. In this article the properties of these spaces under different topological operations have been considered.

Many fundamental results of general topology and functional analysis are obtained in the classes of metric, normed and unitary spaces. Thus, constructive generalizations of these spaces and results obtained for them are of scientific interest.

The most general metrics and norms on topological semifields were considered by Antonovskii, Boltyanskii, and Sarymsakov [1,2]; some papers are devoted to generalizations of metric and norms, in particular, metrics on Banach spaces; see, e.g., [3].

We use notations $R_+ = [0, \infty)$ and $R = (-\infty, +\infty)$. Given an arbitrary cardinal number τ, by R_+^τ and R^τ we denote the topological product of τ copies of the spaces R_+^τ and R^τ, respectively (with the natural topology). On the spaces R_+^τ and R^τ, the operations of addition, multiplication, and multiplication by a scalar, as well as a partial ordering, are defined in a natural way (coordinatewise). The space R^τ transforms into a so-called Tychonoff topological semifield, in which R_+^τ is the positive cone (see [1]).

Topological semifields, metrics and norms on them have arisen in studying function spaces, in particular, ergodic theorems of probability theory; on the contrary, the notions of a τ-metric and a τ-norm studied in this paper have arisen from inside, when studying uniform and locally convex linear topological spaces themselves.

In what follows, we use the following generally known notion (see, e.g., [4]). Let $\{f_\alpha : X \to Y_\alpha : \alpha \in A\}$ be a family of mappings of a set X to a family of sets $\{Y_\alpha : \alpha \in A\}$. The mapping $f : X \to \prod\{Y_\alpha : \alpha \in A\}$

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defined by $fx = \{f_\alpha x : \alpha \in A\}$ is called the diagonal product of the family of mappings $\{f_\alpha : \alpha \in A\}$ and is denoted by $\triangle\{f_\alpha : \alpha \in A\}$.

Let $(X, \mathcal{U})$ be any uniform space, and let $\{\rho_\alpha : \alpha \in A\}$ be any family of pseudometrics on X generating the uniformity $\mathcal{U}$ (see, e.g., [4]) with index set A of cardinality τ. By ρ_τ we denote the diagonal product of the mappings $\rho_\alpha : X_\alpha \times X_\alpha \to R_+$, $\alpha \in A$; i.e., $\rho_\tau = \triangle\{\rho_\alpha : \alpha \in A : X \times X \to R_+^\tau\}$.

Let (L, T) be any locally convex linear topological space, and let $\{P_\beta : \beta \in B\}$ be any family of pseudonorms on L generating the topology T (see, e.g., [5]) with $\mu = |B|$. By $\|\cdot\|$ we denote the diagonal product of the mappings $P_\beta : L \times L \to R_+$, $\beta \in B$; i.e., $\|\cdot\| = \triangle\{\rho_\beta : \beta \in B\} : L \times L \to R_+^\mu$.

The axiomatization of the mappings ρ_τ and $\|\cdot\|$ leads to the notions of a τ-metric and a τ-norm.

Definition 1. Let X be a nonempty set. A mapping $\rho_\tau : X \times X \to R_+^\tau$ is called a τ-metric or a multimetric (if τ is not fixed) on X, and the pair (X, ρ_τ) is called a τ-metric space or a multimetric space, if the following known axioms hold:

(i) $\rho_\tau(x, y) = \theta$ if and only if $x = y$, where θ is the point of the space R_+^τ whose all coordinates are zeros;
(ii) $\rho_\tau(x, y) = \rho_\tau(y, x)$ for all $x, y \in X$;
(iii) $\rho_\tau(x, z) \le \rho_\tau(x, y) + \rho_\tau(y, z)$ for all $x, y, z \in X$.

Any multimetric ρ_τ on a set X generates a uniformity $\mathcal{U}_{\rho_\tau}$ and a topology T_{ρ_τ}. For each neighborhood $O(\theta)$ of the point θ in the space R_+^τ, we set $V_{O(\theta)} = \{(x, y) : \rho_\tau(x, y) \in O(\theta)\}$. The family $\{V_{O(\theta)} : O(\theta)$ ranges over a neighborhood base of θ in the space $R_+^\tau\}$ forms a base of some uniformity $\mathcal{U}_{\rho_\tau}$ on the set X. Let us declare the sets of the form $G(x) = \{y \in X : \rho_\tau(x, y) \in O(\theta)\}$ to be the neighborhoods of $x \in X$; then these sets generate a topology T_{ρ_τ}; moreover, the topology generated by the uniformity $\mathcal{U}_{\rho_\tau}$ coincides with T_{ρ_τ}. Therefore, the topological space (X, T_{ρ_τ}) is Tychonoff (that is, completely regular and Hausdorff).

The following example shows how large the class of τ-metric spaces is.

Example 1. Let $\{(X_\alpha, \rho_\alpha) : \alpha \in A\}$ be any family of metric spaces, and let $\tau = |A|$. Then $\rho_\tau(x, y) = \{\rho_\alpha(x_\alpha, y_\alpha) : \alpha \in A\}$ is a τ-metric on X, where $X = \prod\{X_\alpha : \alpha \in A\}$, $x = \{x_\alpha : \alpha \in A\}$ and $y = \{y_\alpha : \alpha \in A\}$, $x_\alpha, y_\alpha \in X_\alpha$ for each $\alpha \in A$.

The notion of the multimetric space allows to define in natural way an analogue of a contracting mapping so that one can obtain generalization of the Banach fixed point theorem.

Example 2. Let X be a nonempty set of cardinal number m. By $\mathcal{B}_m^\tau$ we denote the product of τ copies of the set X where τ is an infinite cardinal number. Introduce τ-metrics ρ_τ on $\mathcal{B}_m^\tau$ as follows: let $x = \{x_\alpha\}_{\alpha \in A}$ and $y = \{y_\alpha\}_{\alpha \in A}$ be elements of the set $\mathcal{B}_m^\tau$. Denote $\rho_\tau(x, y) = \{a_\alpha\}_{\alpha \in A}$ where

$$a_\alpha = \begin{cases} 1, & \text{if } x_\alpha \ne y_\alpha \\ 0, & \text{if } x_\alpha = y_\alpha. \end{cases}$$

Then $(\mathcal{B}_m^\tau, \rho_\tau)$ is a τ-metrical space. It is a generalization of the Baire metrical space $\mathcal{B}^\tau$ (see [4]).

Definition 2. A mapping $f : (X, \rho_\tau) \to (X, \rho_\tau)$ is called contracting if $\rho_\tau(f(x), f(y)) \le \lambda \cdot \rho_\tau(x, y)$ for all $x, y \in X$ and some λ from $(0, 1)$.

Define the notion of completeness of a τ-metrical space (X, ρ_τ). Let $\xi = \{x_\alpha\}_{\alpha \in A}$ be an arbitrary directivity in the set X where $x_\alpha \in X$ for every $\alpha \in A$. We will say that the directivity $\xi = \{x_\alpha\}_{\alpha \in A}$