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On embeddings of topological groups II

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ABSTRACT

In the present note, for given cardinals τ and μ, $\tau \leq \mu$, we construct a topological group of character τ and of weight $\leq 2^{\mu}$ containing topologically all topological groups of character τ and of weight μ. In particular, if $\tau = \omega$, then there exists a metrizable group of weight $\leq 2^{\mu}$ containing topologically all metrizable groups of weight $\leq \mu$.

1. Introduction

In this note by a topological subgroup of a topological group X we mean an algebraic subgroup considered with the relative topology (therefore we do not assume that a given subgroup is a closed subset of X). Let $\mathbb{S}$ be a class of topological groups. It is said that a topological group K is *universal* in this class if (a) $K \in \mathbb{S}$ and (b) for every $X \in \mathbb{S}$, there exists a topological subgroup of K which is topologically isomorphic to X.

V.V. Uspenskij (see [6]) proved that the group of all self-homeomorphisms of the Hilbert cube with the uniform convergence topology is universal in the class of all separable metrizable topological groups. Such a group is also the group of isometries of the Urysohn universal metric space (see [7]) and the group linear isometries of the Gurarij space is again universal for separable metrizable groups (see [8]). S.A. Shkarin (see [5]) proved that in the class of all separable metrizable topological Abelian groups there exists a universal element. Moreover, he proved that, under GCH, for every uncountable cardinal τ, in the class of all metrizable topological Abelian groups of weight $\leq \tau$ and in the class of all topological Abelian groups

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of weight $\leq \tau$ there are universal elements. However, the problems of the existence of universal elements in the class of all topological groups (see Question 2 of [7]) and in the class of all metrizable topological groups (see Problem 4 of [5]) of a given uncountable weight remain open.

In [2], using the method of construction of so-called Containing Spaces given in [1], a space of a given weight τ containing **continuously** the homeomorphic images of all topological groups of weight at most τ is constructed.

In the present note, for given cardinals τ and μ, $\tau \leq \mu$, we construct a topological group of character τ and of weight 2^μ containing topologically all topological groups of character τ and of weight μ. In particular, if $\tau = \omega$, then there exists a metrizable group of weight $\leq 2^\mu$ containing topologically all metrizable groups of weight $\leq \mu$. From our construction it follows automatically that the above results hold (without GCH) if all considered groups are Abelian.

2. Preliminaries

2.1. On Raikov complete topological groups

Below we recall the notion of Raikov complete topological group. Following the original paper of D.A. Raikov (see [4]) a set Φ of subsets of a topological group X is called *Raikov filter* ("*funnel*" in [4]) if (a) the intersection of any pair of elements of Φ is not empty and (b) for every open neighbourhood U of the identity element of X there exist two elements $M_1, M_2 \in \Phi$ such that $(M_1)^{-1}M_1 \subset U$ and $M_2(M_2)^{-1} \subset U$. Any base of the space X at a point $x \in X$ is a Raikov filter. It is said that a Raikov filter Φ *converges* to a point x of X if each open neighbourhood of x intersects each element of Φ. The topological group X is said to be *Raikov complete* if each Raikov filter converges to a point of X.

2.1.1 Theorem. *Each topological group X has a Raikov completion $\widetilde{X}$, that is, $\widetilde{X}$ is a Raikov complete topological group containing X as a dense topological subgroup. This completion is unique in the sense that if $\widetilde{X}_1$ and $\widetilde{X}_2$ are two Raikov completions of X, then there exists a unique topological isomorphism of $\widetilde{X}_1$ onto $\widetilde{X}_2$, which is identical on X.*

The elements of $\widetilde{X}$ are the classes of *equivalent* (*equal* in [4]) Raikov filters: two such filters Φ_1 and Φ_2 are equivalent if the identity element e^X of X belongs to $Cl_X(MN^{-1})$ for each $M \in \Phi_1$ and each $N \in \Phi_2$. Any point $x \in X$ is identified with the equivalence class containing the Raikov filter Φ_x consisting of open neighbourhoods of x in X. The character and weight of $\widetilde{X}$ coincide with the character and weight of X, respectively.

2.1.2 Notation. An ordinal number is considered as the set of all smaller ordinal numbers, and a cardinal number is considered as the least ordinal with the same cardinality. Therefore, for ordinals δ and τ, the relations $\delta \in \tau + 1$ and $\delta \leq \tau$ are equivalent. By ω we denote the first infinite cardinal. By τ we denote a **fixed infinite cardinal**. The elements of τ will be denoted by δ, ε, and η. By $\mathcal{F}$ we denote the set of all non-empty finite subsets of τ. The symbol $\equiv$ in a relation means that one or both sides of the relation are introduced as new notations.

The following proposition follows from Theorem 2.1.1 and the fact that there exist at most 2^τ pairwise distinct (up to a topological isomorphism) topological groups of both cardinality and weight at most τ.

2.2 Proposition. *The cardinality of any set of pairwise distinct (up to a topological isomorphism) Raikov complete topological groups of weight $\leq \tau$ is $\leq 2^\tau$.*