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The minitightness of products

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We prove that $t_m(X \times Y) \leq t_m(X)t_m(Y)$ if the space Y is locally compact, and that always $t_m(X \times Y) \leq t_m(X)\chi(Y)$, where $t_m(Z)$ is the minitightness (a.k.a. the weak functional tightness) of a space Z.

The minitightness (a.k.a. the weak functional tightness) was defined by A.V. Arhangel'skii in [2]. The reason for an interest to this modification of the tightness is that this cardinal invariant is dual (in the sense of $C_p(X)$) to the *Hewitt number* of a space X, which gives a generalization of realcompactness to arbitrary cardinals. In this article we prove some facts about the minitightness of products and preimages under some special mappings.

All spaces in this article are assumed to be Tychonoff (that is, completely regular Hausdorff). We use terminology and notation as in [3], except that the tightness of a space X is denoted by $t(X)$.

1. Some basic facts

Definition 1.1. ([2]) Let κ be an infinite cardinal, X and Y topological spaces. A function $\phi\colon X \to Y$ is said to be *strictly κ-continuous* if for every subspace A of X such that $|A| \leq \kappa$, the restriction of ϕ to A coincides with the restriction to A of some continuous function $f\colon X \to Y$.

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Proposition 1.2. *If $\phi\colon X \to Y$ and $\psi\colon Y \to Z$ are strictly κ-continuous mappings, then the composition $\psi \circ \phi\colon X \to Z$ is strictly κ-continuous.*

Proof. Let A be a subspace of X with $|A| \le \kappa$. Put $B = \phi(A)$, then $|B| \le \kappa$. By the strict κ-continuity of ψ, there is a continuous $g\colon Y \to Z$ such that $g|B = \psi|B$, and by the strict κ-continuity of ϕ, there is a continuous $f\colon X \to Y$ such that $f|A = \phi|A$. Then the function $h = g \circ f$ is continuous, and the restrictions of h and $\psi \circ \phi$ to A coincide. □

In particular, for every strictly κ-continuous function ϕ on X and a subspace A of X, the restriction of ϕ to A is strictly κ-continuous (as the composition of ϕ with the embedding of A into X).

Definition 1.3. ([2]) The *minitightness* (or the *weak functional tightness*) of a space X is

$$\begin{aligned} t_m(X) = \min\{\kappa : \kappa &\text{ is an infinite cardinal and} \\ &\text{every strictly } \kappa\text{-continuous real function on } X \text{ is continuous}\}. \end{aligned}$$

Let x be a point of a space X. We define the *minitightness of X at x* as

$$\begin{aligned} t_m(x, X) = \min\{\kappa : \kappa &\text{ is an infinite cardinal and} \\ &\text{every strictly } \kappa\text{-continuous real function on } X \text{ is continuous at the point } x\}. \end{aligned}$$

Clearly, always $t_m(X) = \sup\{t_m(x, X) : x \in X\}$.

It was shown in [2] that always $t_m(X) \le t(X)$ and $t_m(X) \le d(X)$; furthermore, $t(X) = \sup\{t_m(Y) : Y \text{ is a subspace of } X\}$, and $t_m(Z) \le t_m(X)$ if Z is the image of X under a quotient mapping.

Proposition 1.4. *The following conditions are equivalent:*

(i) $t_m(X) \le \kappa$;
(ii) *for every space Y, every strictly κ-continuous function from X to Y is continuous;*
(iii) *for every compact space Y, every strictly κ-continuous function from X to Y is continuous;*
(iv) *every strictly κ-continuous function from X to $[0, 1]$ is continuous.*

Proof. (i) $\Longrightarrow$ (ii). Let $\phi\colon X \to Y$ be a strictly κ-continuous function. Then for every real-valued continuous function f on Y, the composition $f \circ \phi$ is a strictly κ-continuous real-valued function on X. From $t_m(X) \le \kappa$ it follows that $f \circ \phi$ is continuous. Thus, the composition of ϕ with every real-valued continuous function on Y is continuous. Since Y is completely regular, this implies that ϕ is continuous.

The implications (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv) are trivial.

(iv) $\Longrightarrow$ (i). Let i be a homeomorphism of $\mathbb{R}$ onto the open interval $(0, 1)$. If $\phi\colon X \to \mathbb{R}$ is a strictly κ-continuous function, then $i \circ \phi$ is a strictly κ-continuous function from X to $[0, 1]$; by the assumption (iv), $i \circ \phi$ is continuous. It follows that ϕ is continuous. □

We say that a subset F of X is *κ-closed* (in X) if for every $B \subset F$ with $|B| \le \kappa$, the closure in X of the set B is contained in F. The tightness of X is less or equal to κ if and only if every κ-closed set in X is closed (this is often taken as the definition of tightness). Note that a set $F \subset X$ is κ-closed in X if and only if for every $A \subset X$ with $|A| \le \kappa$, the intersection $F \cap A$ is closed in A.

We define the *κ-closure* of a set A as $[A]_\kappa = \bigcup\{\bar{B} : B \subset A,\ |B| \le \kappa\}$. It is easy to verify that the κ-closure of any set is κ-closed. We will say that a set A is *κ-dense* in X if $[A]_\kappa = X$. Note that if A is κ-dense in X, and ϕ_1, ϕ_2 are strictly κ-continuous functions of X such that $\phi_1|A = \phi_2|A$, then $\phi_1 = \phi_2$.