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# Computational topology: Isotopic convergence to a stick knot

J. Li [a], T.J. Peters [a,*,1], K.E. Jordan [b], P. Zaffetti [c]

[a] *University of Connecticut, Storrs, CT 06269, USA*
[b] *IBM T.J. Watson Research Center, Cambridge, MA, USA*
[c] *iDevices Inc., Avon, CT 06001, USA*

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ABSTRACT

Computational topology is a vibrant contemporary subfield and this article integrates knot theory and mathematical visualization. Previous work on computer graphics developed a sequence of smooth knots that were shown to converge point wise to a piecewise linear (PL) approximant. This is extended to isotopic convergence, with that discovery aided by computational experiments. Sufficient conditions to attain isotopic equivalence can be determined *a priori*. These sufficient conditions need not be tight bounds, providing opportunities for further optimizations. The results presented will facilitate further computational experiments on the theory of PL knots (also known as stick knots), where this theory is less mature than for smooth knots.



## 1. Introduction & related work

For a positive integer $n$, a Bézier curve [31] of degree $n$ is defined as $\mathbf{B}(t)$, with control points $P_i \in \mathbb{R}^3$ by

$$\mathbf{B}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} P_i, \quad t \in [0,1].$$

The curve $\mathcal{P}$ formed by PL interpolation on the ordered set of points $\{P_0, P_1, \ldots, P_n\}$ is called the *control polygon*. This $\mathcal{P}$ is a PL approximation of $\mathbf{B}$.

The curves considered here will be closed by understanding that $P_0 = P_n$. Furthermore, it is assumed that both the Bézier curves and their control polygons are simple. The focus here is on the isotopic equivalence between a knotted Bézier curve and its PL approximation. These knotted PL approximations are also known as 'stick knots' [1].

* Corresponding author.
*E-mail address:* tpeters@cse.uconn.edu (T.J. Peters).

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A contemporary treatment of knots and molecules [32] provided motivation for much of this work. In particular, remarks on the "... humbling... " status in theoretical understanding of stick knots versus smooth knots stimulated the consideration, here, of Bézier curves that are isotopically equivalent to stick knots.

The term 'molecular movies' was introduced [26] to include the visualization of molecular simulations. There is a specific cautionary example [20] about introducing topological artifacts into a molecular movie. The relevant knots were $4_1$ and the unknot, represented in sufficiently simple format that isotopic equivalences were easily determined by standard methods [23]. Knot visualization software [25] was used to develop that illustrative example.

Those application specific considerations [20] led to the generalizations presented here. The theorem presented here generalizes that example to sufficient conditions for convergence between Bézier curves and their isotopic PL approximations. An *a priori* bound is given on the number of iterations needed to obtain an isotopic approximation.

The preservation of topological characteristics in computational applications is of contemporary interest [2–4,9,10,13,15,17–19,21,24]. The isotopy theorems on knots of finite total curvature are foundational to the work presented here. Sufficient conditions for a homeomorphism between a Bézier curve and its control polygon have been studied [30], while topological differences have also been shown [6,22,31]. Sufficient conditions were given to insure that perturbations of the control points maintain isotopic equivalence of the perturbed splines [5]. There is an example of a PL structure that becomes self-intersecting while the associated Bézier curve remains simple [8]. Recent perspectives on computational topology have appeared [7,14,33].

## 2. Inserting midpoints as control points

The fundamental approximation technique introduced here is dual to many others. The more typical focus is to produce a sequence of PL curves that approximates a given smooth curve. Indeed, these authors have previously published such results [17]. The duality here is to create a sequence of smooth knots that converge to a PL knot, achieving isotopic equivalence within the sequence. The use of smoothness here is understood to be $C^\infty$, with a possible exception at the point $\mathbf{B}(0) = \mathbf{B}(1)$. This technique has been called *collinear insertion* [20], where an example was presented with 8 initial control points (inclusive of equality of the initial and final control points). That control polygon was the knot $4_1$ and its associated Bézier curve was the unknot. With 4 iterations of collinear insertion, the stick and smooth knots were isotopic.

**Definition 2.1.** Consider the control polygon $\mathcal{P}$. To avoid trivial cases, it is assumed that, for $i > 0$,

- $P_i \neq P_i*$ for any $i \neq i*$,
- only two such $P_i \neq P_i*$ can be collinear and
- $n \geq 4$.

Sequences of control polygons and Bézier curves will be generated by letting $\mathcal{P}^{(0)} = \mathcal{P}$ and generating $\mathcal{P}^{(1)}$ from $\mathcal{P}^{(0)}$ by adding the midpoint of each edge of $\mathcal{P}^{(0)}$. For $j \geq 0$, similarly generate $\mathcal{P}^{(j+1)}$ by the insertion of midpoints between all of the control points of $\mathcal{P}^{(j)}$. For each $j$, the corresponding Bézier curve will be denoted as $\mathbf{B}^{(j)}$. Note that all $\mathcal{P}^{(j)}$'s are isotopic under the trivial identification map.

## 3. Convergence theorem

The primary convergence result relies upon a previously published theorem [13, Theorem 4.2] on rectifiable [28] graphs of finite total curvature [27]. This central theorem is quoted, below, after definition of a key notion of closeness [13].