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Every simple compact semiring is finite

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ABSTRACT

A Hausdorff topological semiring is called simple if every non-zero continuous homomorphism into another Hausdorff topological semiring is injective. Classical work by Anzai and Kaplansky implies that any simple compact ring is finite. We generalize this result by proving that every simple compact semiring is finite, i.e., every infinite compact semiring admits a proper non-trivial quotient.

1. Introduction

In this note we study simple Hausdorff topological semirings, i.e., those where every non-zero continuous homomorphism into another Hausdorff topological semiring is injective. A compact Hausdorff topological semiring is simple if and only if its only closed congruences are the trivial ones. The structure of simple compact rings is well understood: a classical result due to Kaplansky [7] states that every simple compact Hausdorff topological ring is finite and thus – by the Wedderburn–Artin theorem – isomorphic to a matrix ring $\mathrm{M}_n(\mathbb{F})$ over some finite field $\mathbb{F}$. In particular, it follows that any compact field is finite. We note that Kaplansky's result may as well be deduced from earlier work of Anzai [1], who proved that every compact Hausdorff topological ring with non-trivial multiplication is disconnected and that moreover every compact Hausdorff topological ring without left (or right) total zero divisors is profinite, i.e., representable as a projective limit of finite discrete rings. Of course, a generalization of the mentioned results by Anzai cannot be expected for general compact semirings: in fact, there are numerous examples of non-zero connected compact semirings with multiplicative unit, which in particular cannot be profinite. However, in the present paper, we extend Kaplansky's result and show that any simple compact Hausdorff topological semiring is

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finite (Theorem 4.6). Hence, the classification of finite simple semirings applies, which has been established in [10].

2. Semirings

In this section we briefly recall several elementary concepts concerning semirings. For a start let us fix some general terminology. We assume the reader to be familiar with classical algebraic structures or algebras, such as semigroups, monoids, groups, and rings, as well as the related concepts of subalgebras, homomorphisms, and product algebras. If A is any algebra, then a *congruence* on A is an equivalence relation on A constituting a subalgebra of the product algebra $A \times A$.

Let R be a *semiring*, i.e., an algebra $(R, +, \cdot, 0)$ satisfying the following conditions:

— $(R, +, 0)$ is a commutative monoid and $(R, \cdot)$ is a semigroup,
— $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$,
— $0 \cdot x = x \cdot 0 = 0$ for every $x \in R$.

A *subsemiring* of R is a subset $A \subseteq R$ such that A is a submonoid of the additive monoid of R and a subsemigroup of the multiplicative semigroup of R. Similarly, a *homomorphism* from R into another semiring S is a map $h\colon R \to S$ such that h is both a homomorphism from the additive monoid of R to that of S and from the multiplicative semigroup of R to that of S. Clearly, an equivalence relation θ on R is a congruence on R if and only if θ is a congruence on the additive monoid of R and the multiplicative semigroup of R. As usual, an *ideal* of R is a submonoid A of the additive monoid of R such that $RA \cup AR \subseteq A$. Moreover, a subset $A \subseteq R$ is called *subtractive* if the following holds:

$$\forall x \in R \, \forall a \in A\colon \; x + a \in A \Longrightarrow x \in A.$$

Subtractive ideals are closely related to the following congruences due to Bourne [3], as the subsequent basic lemma reveals.

Lemma 2.1. *([3]) Let R be a semiring and let A be an ideal of R. Then*

$$\kappa_A := \{(x, y) \in R \times R \mid \exists a, b \in A\colon \; x + a = y + b\}$$

is a congruence on R. Furthermore, A is subtractive if and only if $A = [0]_{\kappa_A}$.

Let us turn our attention towards naturally ordered semirings. To this end, let $(M, +, 0)$ be a commutative monoid. Notice that the relation given by

$$x \leq y \;:\Longleftrightarrow\; \exists z \in M\colon \; x + z = y \qquad (x, y \in M)$$

is a preorder, i.e., $\leq$ is reflexive and transitive. We say that M is *naturally ordered* if the preorder $\leq$ is anti-symmetric, which means that $(M, \leq)$ is a partially ordered set with least element 0. Now let R be a semiring and consider the preorder $\leq$ defined as above with regard to the additive monoid of R. It is straightforward to check that

$$\forall x, x', y, y' \in R\colon \; x \leq x', \, y \leq y' \;\Longrightarrow\; x + y \leq x' + y', \; xy \leq x'y'.$$

We say that R is *naturally ordered* if its additive monoid is naturally ordered. Such a semiring R is called *bounded* if the partially ordered set $(R, \leq)$ is bounded, in which case we denote the greatest element by ∞. One may easily deduce the following observations.