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On "ternary" density points in Cantor space

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ABSTRACT

We prove inclusions between the density topology on the real line, the density topology on the Cantor set and (defined in this article) "ternary" density topology on the Cantor set via standard "almost injection" $\phi: 2^{\mathbb{N}} \to \mathbb{R}$. We also prove some properties of the "ternary" density topology.

1. Preliminaries

This article is devoted to find the correspondence between the density topology on the real line and the density topology on the Cantor set via the standard continuous almost everywhere one-to-one function $\phi: 2^{\mathbb{N}} \to \mathbb{R}$. This idea was inspired by the articles [2] and [3] where special ideal was defined by using the density topology and there was an attempt to prove some properties of this ideal in the real line case (see [2]) by the Cantor set case (see [3]) using the hypothetical correspondence between the density topology on the real line and on the Cantor set. In this article we will show that there are some inclusions between these topologies, also we define special "ternary" density topology to close the diagram.

2. Definitions

We denote the set of "rational numbers" of $2^{\mathbb{N}}$ by Q, so $Q = \{x \in 2^{\mathbb{N}}: \exists_{n_0} \forall_{n \geq n_0} x(n) = 0\}$.

Moreover, define $\tilde{Q} = \{x \in 2^{\omega}: \exists_{n_0} \forall_{n > n_0} x(n) = x(n_0)\}$. We have $\tilde{Q} = Q \cup (Q + \mathbf{1})$ (where $\forall_n \mathbf{1}(n) = 1$).

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Let $\phi: 2^{\mathbb{N}} \to \mathbb{R}$ be defined by the formula $\phi(x) = \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}}$. Then $\phi \restriction 2^{\mathbb{N}} \setminus Q$ is an injection.

We denote by τ_d the classical density topology on the real line. Notice that the classical density topology was defined for the case of the real line, but it is well known that we can define an analogous notion also in the case of the Cantor set. Namely, if $A \subseteq 2^\omega$ is a measurable set, then define the set of the density points as follows.

Definition 1 *([4], Exercise 17.9).*

$$x \in \Phi_{2^\omega}(A) \iff \lim_{n\to\infty} \frac{\mu([x\restriction n] \cap A)}{\mu([x\restriction n])} = 1.$$

Such a notion for the Cantor set has the same properties as in the case of the density topology on the real line. Indeed, the collection $\tau_d(2^\omega)$ of all measurable sets $A \subseteq 2^\omega$ such that $A \subseteq \Phi_{2^\omega}(A)$, is a topology.

It is an easy observation that the image of a set open in the topology $\tau_d(2^{\mathbb{N}})$ by the function ϕ need not be open in the density topology τ_d. Indeed, we have $\phi[[0]] = [0, \frac{1}{2}]$. Therefore, the natural question arises as to whether one can find another kind of natural density topology in $2^{\mathbb{N}}$ such that the image of set open in this topology again by the function ϕ is always open in the topology τ_d. The aim of the next theorem is to answer this question in the affirmative.

Definition 2. For any $n \in \mathbb{N}$ let $\langle 2^n, <\rangle$ be the set of all binary sequences of length n with lexicographical order. For any $t \in 2^n$, if there exists some $i = 1, 2, \ldots, n$ such that $t_i = 0$, we define t^+ as a successor of t.

For any $t \in 2^n$, if there exists some $i = 1, 2, \ldots, n$ such that $t_i = 1$, we define t^- as such element of 2^n that t is the successor of t^-. Moreover, we define $(1, \ldots, 1)^+ = (0, \ldots, 0)$ and $(0, \ldots, 0)^- = (1, \ldots, 1)$.

We shall modify the definition of a density point:

Definition 3. For any measurable $A \subseteq 2^{\mathbb{N}}$ a point $x \in 2^{\mathbb{N}}$ is said to be the symmetric density point of A if and only if

$$\lim_{n\to\infty} \frac{\mu\Big(\big([(x\restriction n)^-] \cup [x\restriction n] \cup [(x\restriction n)^+]\big) \cap A\Big)}{3\cdot\mu([x\restriction n])} = 1.$$

The set of all symmetric density points of A we shall denote by $\Phi'_{2^{\mathbb{N}}}(A)$.

Theorem 2.1. *The following inclusions hold for any measurable $A \in 2^{\mathbb{N}}$:*

$$\Phi'_{2^{\mathbb{N}}}(A) \subseteq \phi^{-1}[\Phi(\phi[A] + \mathbb{Z})] \subseteq \Phi_{2^\omega}(A).$$

Proof of the first inclusion. Choose $n_0 \in \omega$ such that for each $n > n_0$ we have

$$\frac{\mu\Big(\big([x\restriction n] \cup [(x\restriction n)^+] \cup [(x\restriction n)^-]\big) \cap A\Big)}{3\cdot\mu([x\restriction n])} > 1 - \epsilon.$$

In order to simplify the proof, we will use the following notation:

Definition 2.2. For any $t \in 2^{<\omega}$ let us denote $D_t = \phi[[t]]$ and

$$D_t^+ = \begin{cases} \phi[[t^+]] & \text{if } t \neq 1\restriction|t| \\ \phi[[t^+]] + 1 & \text{if } t = 1\restriction|t| \end{cases}$$