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Zero-dimensional almost 1–1 extensions of odometers from graph coverings

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A R T I C L E I N F O

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A B S T R A C T

It is well known that the one- or two-sided Toeplitz flows are characterised as the symbolic almost 1–1 extensions of infinite odometers. The two-sided Toeplitz flows are also characterised as the Bratteli–Vershik systems with the equal path number property. Nevertheless, for one-sided systems, the Bratteli–Vershik way is not suitable as a combinatorial representation. We summarise one- or two-sided almost 1–1 extensions of infinite odometers by the graph covering that Gambaudo and Martens (2006) presented. They used the inverse limit of a certain kind of sequences of finite directed graphs. When Gjerde and Johansen (2000) characterised the two-sided Toeplitz flows, they used the notion of the equal path number property. The notion we employ is the translated equal period property that implies that all the circuits of graphs have equal period in the way of graph coverings of Gambaudo and Martens. In our summary, we also characterise the one-sided Toeplitz flows by these graph coverings. As an application, we show that the natural extension of a one-sided Toeplitz flow is Toeplitz. We also summarise the link between the general Bratteli–Vershik representations and the graph coverings that Gambaudo and Martens gave. If we consider the expansiveness, taking the natural extension is very significant. We also show that the family of natural extensions of inverse limits of their coverings with the equal period property coincides with the two-sided zero-dimensional homeomorphisms that are almost 1–1 extensions of odometers. Gjerde and Johansen's original aim highlighted the difficulty in finding a condition of expansiveness for ordered Bratteli diagrams. Sugisaki (2001) responded by deriving a sufficient condition for expansiveness; we extend this condition using the graph coverings given by Gambaudo and Martens. We also discuss some relation between the expansiveness of the natural extensions of inverse limits of graph coverings and the positive expansiveness of the inverse limits. As an application, we show that the topological rank of a one-sided minimal subshift is not greater than its natural extension.

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1. Introduction

According to Gjerde and Johansen [6, Theorem 8], the two-sided Toeplitz flows can be characterised as expansive Bratteli–Vershik systems of properly ordered Bratteli diagrams with the equal path number property. If we consider one-sided Toeplitz flows, the Bratteli–Vershik systems are not suitable. For all zero-dimensional minimal continuous surjections, Gambaudo and Martens [5] derived a representation using certain sequences of finite directed graphs, which we refer to as 'graph coverings'. With this representation, we summarise some results that are related to the Bratteli–Vershik representation, Toeplitz flows (both one-sided and two-sided), zero-dimensional almost 1–1 extensions of odometers, and the graph coverings given by Gambaudo and Martens. First, in §2, following the work of Downarowicz [2], we recall some basic facts about topological dynamical systems. We also provide the basic definitions and notation of our graph coverings. Second, we link the general Bratteli–Vershik representations to our graph coverings in §3. Third, we define the graph coverings given by Gambaudo and Martens in §4 via Definitions 4.1 and 4.2. The rest of §4 is devoted to the study of zero-dimensional almost 1–1 extensions of odometers by the ways of graph coverings. With these basic results, we deal with a concrete representation of one-sided Toeplitz flows in §5 by the graph coverings. In §6, we consider the relation between the graph coverings and the expansiveness. In this section, we also consider the positive expansiveness from the view of graph coverings.

It is natural for us to focus our discussion on Toeplitz flows. They were also characterised as minimal subshifts that are almost 1–1 extensions of infinite odometers (see the Introduction of Downarowicz and Durand [3]). These two characterisations of the Toeplitz flows are related. We attempt to characterise the zero-dimensional almost 1–1 extensions of infinite odometers using the graph coverings given by Gambaudo and Martens. The graph coverings method is combinatorial by definition, and the proofs can sometimes be constructive. We use such constructive proofs. Downarowicz [2] surveyed a number of studies on Toeplitz flows that are not only two-sided, but also one-sided as the almost 1–1 extensions of infinite odometers. His work also considered the almost 1–1 extensions of odometers themselves. Nevertheless, he, in [2], did not include Bratteli–Vershik representations for the systems he surveyed. Perhaps, it had a conflict with his way to summarise not only two-sided homeomorphic systems but also one-sided systems. According to the graph covering that Gambaudo and Martens gave, we can deal with both at the same time. We formulate a link between the zero-dimensional almost 1–1 extensions of odometers and graph coverings using a property that we call the equal period property (see Definition 4.16). We characterise the zero-dimensional almost 1–1 extensions of odometers using the graph coverings given by Gambaudo and Martens (see Theorem 4.19). We also characterise two-sided Toeplitz flows as the expansive natural extensions of the inverse limit of the graph coverings with the equal period property (see Theorem 4.23). Later, in §6, we discuss some relation between the expansiveness of the natural extension of an inverse limit of a graph covering and the positive expansiveness of the inverse limits. Williams [14] has presented a concrete fundamental study of Toeplitz flows. Using this, we give a concrete representation of one-sided Toeplitz flows (see Proposition 5.5 and Theorem 5.9). We then show that the natural extension of a one-sided Toeplitz flow is Toeplitz in Corollary 5.8. This result is made possible by the work of Downarowicz [2].

In the characterisations using graph coverings, as well as those using Bratteli–Vershik systems, expansiveness and positive expansiveness have always been implicitly and explicitly questioned. In [6], Gjerde and Johansen posed an open problem at the end of [6, §2], stating that it was difficult to find conditions of expansiveness under certain conditions. Sugisaki [13, Theorem 4.1] attempted to solve this problem via operations on Bratteli diagrams, and eventually gave a very simple sufficient condition under which Bratteli–Vershik systems are expansive. In §6, we present a simple sufficient condition that extends his result (see Definition 6.3 and Theorem 6.4). Downarowicz and Maass [4] gave a totally different condition by which Bratteli–Vershik systems are expansive, showing that minimality with the finite topological rank $K > 1$ brings about expansiveness. Following this work, we defined a topological rank for zero-dimensional minimal continuous surjections [11], and showed that this rank coincides with that of Downarowicz and Maass for