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Not every transitively D-space is D

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ABSTRACT

In 2008, Liang-Xue Peng defined a weakening of the D-space property known as the transitively D property. He asked whether or not every transitively D-space is D. We show that three different examples of linearly Lindelöf, non-Lindelöf spaces are transitively D but not D, answering Peng's question in the negative.

1. Introduction

The concept of a D-space was first introduced by E. van Douwen in 1979 [9] from the notion of a neighbourhood assignment on a topological space $(X, \mathcal{T})$. A neighbourhood assignment is a function $\phi : X \to \mathcal{T}$ such that $x \in \phi(x)$ for all $x \in X$. A kernel for ϕ is a subset $Y \subset X$ such that $\bigcup\{\phi(x) : x \in Y\} = X$. X is a D-space if any neighbourhood assignment ϕ has a closed discrete kernel. The notion of a D-space is a natural covering property, yet there are very few known theorems relating more well known covering properties to it. While it is an easy observation that every compact space is D, it is unknown even whether every regular Lindelöf space is D. Much of the research related to D-spaces has been centred around this question, though little progress has been made. There is an example due to D. Soukup and P. Szeptycki of a Hausdorff space which is Lindelöf but not D, but this space is not regular [8]. On the other hand, there are no known examples of non-D spaces satisfying a covering property even as weak as submetalindelöfness [2]. There is an example of space which is weakly submetalindelöf but not D, found by van Douwen and H.H. Wicke [10].

One way to approach problems about D-spaces is to weaken the D property by restricting the types of allowed neighbourhood assignments, in order to more readily prove theorems relating neighbourhood

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assignment kernels to classical covering properties. Two weakenings of this form are the transitively D and linearly D properties [7,3]. A space is linearly D if every linear neighbourhood assignment (that is, a neighbourhood assignment which is linearly ordered with respect to inclusion) has a closed discrete kernel. The definition of transitively D is slightly more complicated. We say that a neighbourhood assignment $\phi : X \longrightarrow \mathcal{T}$ is transitive if whenever $y \in \phi(x)$, then $\phi(y) \subset \phi(x)$. A space is transitively D if every transitive neighbourhood assignment has a closed discrete kernel. These properties are much easier to work with than the D property. For example, it is possible to show that any metalindelöf space is both linearly D and transitively D [7,3].

In [7], L-X. Peng asked whether every transitively D space is D. A positive answer to Peng's question would solve many of the major unanswered questions about D-spaces, including the issue of whether every Lindelöf space is D. Every linearly D space is transitively D, so it is logical to look at linearly D, non-D spaces for counterexamples to this question. In this vein, G. Gruenhage [2] suggested that known examples of linearly Lindelöf non-Lindelöf spaces might possess the property of being transitively D but not D. A space X is linearly Lindelöf if any open cover of X that can be linearly ordered with respect to inclusion has a countable subcover. Equivalently, X is linearly Lindelöf if any uncountable subset $Y \subset X$ of regular cardinality has a complete accumulation point. Any linearly Lindelöf space X is linearly D [3], but cannot be D since any such space has countable extent but uncountable Lindelöf number. We show that three different spaces of this nature are in fact transitively D, answering Peng's question.

In order to more easily identify what types of spaces might be transitively D, we characterize these spaces using the following property.

Definition 1.1. Let $\mathcal{U}$ be an open cover for a topological space X. We say that $\mathcal{U}$ is interior-preserving if for any $\mathcal{V} \subset \mathcal{U}$, $\bigcap \mathcal{V}$ is open in X. A neighbourhood assignment ϕ is interior-preserving if the collection $\{\phi(x) : x \in X\}$ is an interior-preserving open cover.

In [4], H. Junnila showed that every transitive neighbourhood assignment could be obtained from an interior-preserving open cover, and conversely, every interior preserving open cover could give rise to a transitive neighbourhood assignment by taking appropriate intersections of sets. We can use this idea to restate the transitively D property as so:

Lemma 1.2. *A topological space X is transitively D if and only if every interior-preserving neighbourhood assignment has a closed discrete kernel.*

Proof. For the "if" direction, suppose that $\{\phi(x) : x \in X\}$ is a transitive neighbourhood assignment on X. Let $Y \subset X$, and suppose that $x \in \bigcap\{\phi(y) : y \in Y\}$. Then $x \in \phi(y)$ for all $y \in Y$, so $\phi(x) \subset \phi(y)$ for all $y \in Y$ and thus $\phi(x) \subset \bigcap\{\phi(y) : y \in Y\}$. Thus $\bigcap\{\phi(y) : y \in Y\}$ is open, and since Y was arbitrary, $\{\phi(x) : x \in X\}$ is interior-preserving. Thus we can find a closed discrete kernel for ϕ, so X is transitively D. For the "only if" direction, suppose that X is transitively D, and let ϕ be a neighbourhood assignment. Define a new neighbourhood assignment ψ by letting $\psi(x) = \cap\{\phi(y) : x \in \phi(y)\}$. Then given any $x, y \in X$, if $x \in \psi(y)$, then $x \in \phi(z)$ for all z such that $y \in \phi(z)$, so $\psi(x) = \cap\{\phi(z) : x \in \phi(z)\} \subset \cap\{\phi(z) : y \in \phi(z)\} = \psi(y)$. Thus ψ is a transitive neighbourhood assignment, so we can find a closed discrete subset D such that $\bigcup\{\psi(x) : x \in D\} = X$. But $\psi(x) \subset \phi(x)$ for all $x \in X$, so $\bigcup\{\phi(x) : x \in D\} = X$ and so ϕ has a closed discrete kernel. □

2. Examples of transitively D, non-D spaces

Example 2.1. The first space that we consider was discovered independently by Gruenhage and R. Buzyakova (see [1]). Let D be the two-point space $\{0, 1\}$ with the discrete topology and let $X^* = D^{\omega_\omega}$. For a point