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Loop homological invariants associated to real projective spaces

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Let A be a based subspace of Y. Under the assumptions that Y is path-connected and that the reduced diagonal map of A induces the zero map in all mod 2 reduced homology groups, we compute a formula for the mod 2 reduced Poincaré series of the loop space $\Omega((A \wedge \mathbb{RP}^\infty) \cup_{A\wedge\mathbb{RP}^1} (Y \wedge \mathbb{RP}^1))$. Here $\mathbb{RP}^\infty$ and $\mathbb{RP}^1$ denote the infinite real projective space and the real projective line respectively.

1. Introduction

A general problem in algebraic topology is to compute the homology of a loop space. More precisely, for coefficients in a field K and given a pointed space X, one can ask to determine the homology $H_*(\Omega X; K)$ as a Hopf algebra. Here the multiplication of $H_*(\Omega X; K)$ is induced by multiplication of loops $\mu : \Omega X \times \Omega X \to \Omega X$, while the comultiplication of $H_*(\Omega X; K)$ is induced by the diagonal map $\Delta_{\Omega X} : \Omega X \to \Omega X \times \Omega X$. In the case where $X = \Sigma Y$ is the suspension of a path-connected space Y, this was determined by Bott–Samelson [1]. They proved that $H_*(\Omega\Sigma Y; K)$ is isomorphic as a Hopf algebra to the tensor algebra $T(\tilde{H}_*(Y; K))$ of the reduced homology of Y, with the comultiplication of the tensor algebra determined on generators by the comultiplication of $\tilde{H}_*(Y; K)$.

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In the case where K is of characteristic zero, say K is the field $\mathbb{Q}$ of rational numbers, Milnor–Moore proved that $H_*(\Omega X;\mathbb{Q})$ is isomorphic as a Hopf algebra to the universal enveloping algebra $U(\pi_*(\Omega X)\otimes\mathbb{Q})$ where the Lie bracket on $\pi_*(\Omega X)\otimes\mathbb{Q}$ is given by the Samelson product [12]. However, the structure of Hopf algebras is less understood when K is of characteristic a prime p (although see [7,8]). In this article, we will be interested in the case where $p = 2$.

When unable to determine the homology $H_*(\Omega X; K)$ as a Hopf algebra, one can forget the multiplication and comultiplication, asking only to compute the reduced Poincaré series of ΩX. Recall that, for W a pointed space all of whose homology groups $H_q(W; K)$ are finite-dimensional K-vector spaces, its qth *reduced Betti number* $\tilde{b}_q(W; K)$ is the dimension of the K-vector space $\tilde{H}_q(W; K)$ and its *reduced Poincaré series* is the ordinary generating function of its reduced Betti numbers, namely the formal power series $\tilde{P}(W; K) := \sum_{q\geq 0} x^q \tilde{b}_q(W; K)$. For example, the Bott–Samelson theorem described above implies that, if Y is a path-connected space all of whose homology groups $\tilde{H}_q(Y; K)$ are finite-dimensional, then

$$\tilde{P}(\Omega\Sigma Y; K) = \frac{\tilde{P}(Y; K)}{1 - \tilde{P}(Y; K)}. \tag{1}$$

For K of arbitrary characteristic, a standard strategy to compute the loop space homology $H_*(\Omega X; K)$ is to consider the Serre spectral sequence [13] and the Eilenberg–Moore spectral sequence [4] associated to the path-loop fibration $\Omega X \to PX \to X$.

Another strategy is to construct a topological monoid whose underlying space is ΩX. Again considering the example where $X = \Sigma Y$ is the suspension of a path-connected space Y, James proved that the reduced free topological monoid $J[Y]$ has homotopy type $\Omega\Sigma Y$ and used the associated word filtration to prove the suspension splitting $\Sigma\Omega\Sigma Y \simeq \bigvee_{s\geq 1} \Sigma Y^{\wedge s}$ [9]. This gives another proof of (1). The idea behind this strategy is to exploit the strictly associative multiplication of the constructed topological monoid. This strict associativity is easier to exploit than the homotopy coherent associativity of the multiplication of loops $\mu : \Omega X \times \Omega X \to \Omega X$ which gives ΩX its A_∞-space structure [15].

In addition, one can apply techniques from simplicial homotopy theory. For X a reduced simplicial set, Kan constructed a free simplicial group GX whose underlying space is ΩX [10]. Taking K to be the field $\mathbb{F}_2$ of two elements, Bousfield–Curtis used Kan's construction to develop a spectral sequence which converges to $H_*(\Omega X; \mathbb{F}_2)$ when X is simply connected [2]. A consequence of their work is the following result (see Proposition 4.1 below): If X is a simply-connected pointed space whose reduced diagonal map $\overline{\Delta}_X : X \to X \wedge X$ induces the zero map in all mod 2 reduced homology groups, then

$$\tilde{P}(\Omega X; \mathbb{F}_2) = \frac{\tilde{P}(X; \mathbb{F}_2)}{x - \tilde{P}(X; \mathbb{F}_2)}. \tag{2}$$

Here the reduced diagonal map $\overline{\Delta}_X$ is the composite $X \xrightarrow{\Delta_X} X \times X \twoheadrightarrow X \wedge X$ of the diagonal map followed by the standard projection to the self-smash product. In particular, (2) holds when X is a simply-connected co-H-space. This is a generalization of (1) in the case where $K = \mathbb{F}_2$.

In this article, we compute the mod 2 reduced Poincaré series for a certain loop space which is the underlying space of a simplicial group construction of Carlsson. This culminates work beginning from Carlsson [3] and followed by the first and third authors [18,5,6]. Let $\mathbb{RP}^1$ denote the real projective line, regarded as a subspace of the infinite real projective space $\mathbb{RP}^\infty$. In terms of the standard CW complex structure on $\mathbb{RP}^\infty$, the subspace $\mathbb{RP}^1$ is the bottom cell. Note that both these real projective spaces are Eilenberg–MacLane spaces, namely $\mathbb{RP}^1 = K(\mathbb{Z}, 1)$ and $\mathbb{RP}^\infty = K(\mathbb{Z}/2, 1)$. In particular, $\mathbb{RP}^1 \simeq S^1$.

Theorem 1.1. *Let $A \hookrightarrow Y$ be a based inclusion of pointed spaces, both of whose mod 2 homology groups are finite-dimensional. If Y is path-connected and the map $(\overline{\Delta}_A)_* : \tilde{H}_*(A; \mathbb{F}_2) \to \tilde{H}_*(A \wedge A; \mathbb{F}_2)$ in mod 2 reduced*