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On cardinal sequences of LCS spaces

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ABSTRACT

If α is an ordinal, we denote by $\mathcal{C}(\alpha)$ the class of all cardinal sequences of length α associated with locally compact scattered spaces (its precise definition is given in Section 1). In this paper, we present a general construction of locally compact scattered spaces with a large top. As consequences of this construction we obtain the following results:

(1) If κ is a singular cardinal of cofinality ω, then $\langle\kappa\rangle_\kappa \frown \langle\kappa^\omega\rangle \in \mathcal{C}(\kappa+1)$.
(2) If κ is an inaccessible cardinal, then $\langle\kappa\rangle_\kappa \frown \langle\kappa^\kappa\rangle \in \mathcal{C}(\kappa+1)$.
(3) If GCH holds, then for any infinite cardinal κ we have $\langle\kappa\rangle_\kappa \frown \langle\kappa^{\mathrm{cf}(\kappa)}\rangle \in \mathcal{C}(\kappa+1)$.

Also, we prove that if κ is a singular cardinal of cofinality ω, then for every cardinal λ such that $\kappa < \lambda \leq \kappa^\omega$ we have that $\langle\kappa\rangle_\kappa \frown \langle\lambda\rangle_{\omega_2} \in \mathcal{C}(\kappa+\omega_2)$.

1. Introduction

All spaces in this paper are assumed to be Hausdorff. Recall that a space X is *scattered*, if every non-empty subspace of X has an isolated point. Then, by an LCS *space* we mean a locally compact, Hausdorff and scattered space. If X is an LCS space and α is an ordinal, we define the α-th Cantor–Bendixson *level* of X by $I_\alpha(X) =$ the set of isolated points of $X \setminus \bigcup\{I_\beta(X) : \beta < \alpha\}$. The *height* of an LCS space X is defined by $\mathrm{ht}(X) =$ the least ordinal α such that $I_\alpha(X) = \emptyset$. And the *reduced height* of X is defined by $\mathrm{ht}^-(X) =$ the least ordinal α such that $I_\alpha(X)$ is finite. Clearly, one has $\mathrm{ht}^-(X) \leq \mathrm{ht}(X) \leq \mathrm{ht}^-(X) + 1$. The *cardinal sequence* of X, in symbols $\mathrm{CS}(X)$, is defined as the sequence formed by the cardinalities of the infinite Cantor–Bendixson levels of X, i.e.

$$\mathrm{CS}(X) = \langle |I_\alpha(X)| : \alpha < \mathrm{ht}^-(X)\rangle.$$

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Many authors have studied the possible sequences of infinite cardinals that can arise as the cardinal sequence of an LCS space. It was proved by Juhász and Weiss that if $f = \langle \kappa_\alpha : \alpha < \omega_1 \rangle$ is a sequence of infinite cardinals, then f is the cardinal sequence of an LCS space iff $\kappa_\beta \leq \kappa_\alpha^\omega$ for every $\alpha < \beta < \omega_1$ (see [7]). The situation becomes, however, more complicated when we consider cardinal sequences of length greater than ω_1. Nevertheless, a characterization under GCH for cardinal sequences of length $< \omega_2$ was obtained by Juhász, Soukup and Weiss in [5]. However, no characterization is known for cardinal sequences of length ω_2. We refer the reader to the survey papers [1] and [14] for a wide list of results on cardinal sequences as well as examples and basic facts. In particular, it is known that the notion of an LCS space is forcing indestructible and that results on cardinal sequences for LCS spaces can be directly translated into the context of superatomic Boolean algebras.

If α is an ordinal, we put $\mathcal{C}(\alpha) = \{\mathrm{CS}(X) : X \text{ is an LCS space and } \mathrm{ht}^-(X) = \alpha\}$. If κ is an infinite cardinal and α is an ordinal, we denote by $\langle \kappa \rangle_\alpha$ the cardinal sequence $\langle \kappa_\beta : \beta < \alpha \rangle$ where $\kappa_\beta = \kappa$ for $\beta < \alpha$. If f and g are sequences of infinite cardinals, we denote by $f \frown g$ the concatenation of f with g.

Assume that X is an LCS space. We define the *width* of X by $\mathrm{wd}(X) = \sup\{|I_\alpha(X)| : \alpha < \mathrm{ht}(X)\}$. If κ is an infinite cardinal, we say that X is *κ-thin-tall*, if $\kappa \leq \mathrm{wd}(X) < |\mathrm{ht}(X)|$. And if γ is an infinite ordinal, we say that X is a *γ-thin-thick* space, if $\mathrm{CS}(X) = \langle \kappa_\alpha : \alpha \leq \gamma \rangle$ where $\kappa_\alpha \leq |\gamma|$ for $\alpha < \gamma$ and $\kappa_\gamma \geq |\gamma|^+$.

It is well known that $\langle \omega \rangle_\alpha \in \mathcal{C}(\alpha)$ for every ordinal $\alpha < \omega_2$ (see [6] or [7]) and that it is relatively consistent with ZFC that $\langle \omega \rangle_\alpha \in \mathcal{C}(\alpha)$ for every ordinal $\alpha < \omega_3$ (see [2] and [12]). Also, it was proved by Just in [8] that in the Cohen model, $\langle \omega \rangle_{\omega_2} \notin \mathcal{C}(\omega_2)$ and $\langle \omega \rangle_{\omega_1} \frown \langle \omega_2 \rangle \notin \mathcal{C}(\omega_1 + 1)$. In addition, it was proved by Baumgartner in [2] that $\langle \omega_1 \rangle_{\omega_1} \frown \langle \omega_2 \rangle \notin \mathcal{C}(\omega_1 + 1)$ in the Mitchell model. On the other hand, it was shown in [9] that if V = L holds then for every regular cardinal κ, $\langle \kappa \rangle_{\kappa^+} \in \mathcal{C}(\kappa^+)$ and $\langle \kappa \rangle_\kappa \frown \langle \kappa^+ \rangle \in \mathcal{C}(\kappa + 1)$. And recently, as a consequence of the main theorem of [13], we obtained that if κ, λ are specific infinite cardinals with $\kappa < \lambda$ and κ regular, then it is consistent that $\langle \kappa \rangle_{\kappa^+} \frown \langle \lambda \rangle \in \mathcal{C}(\kappa^+ + 1)$.

However, for κ a singular cardinal, no result is known on the existence of κ-thin-tall spaces and very little is known on the existence of κ-thin-thick spaces. In relation to these problems, it was proved in [4] that if κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$ then $\langle \kappa \rangle_\kappa \frown \langle \kappa^+ \rangle_{\kappa^+} \in \mathcal{C}(\kappa^+)$. Then, by using the refinement of Prikry forcing due to Magidor (see [3, Chapter 36]) and the well-known fact that the notion of an LCS space is forcing indestructible, we obtain as a corollary of this result that if it is consistent that there is a measurable cardinal, then it is consistent that $\langle \aleph_\omega \rangle_{\aleph_\omega} \frown \langle \aleph_\omega^+ \rangle_{\aleph_\omega^+} \in \mathcal{C}(\aleph_\omega^+)$.

Then, we shall show here the following general result on the existence of κ-thin-thick spaces: if κ is an infinite cardinal and $\delta < \kappa^+$ is a limit ordinal such that $\kappa^{<\mathrm{cf}(\delta)} = \kappa$, then $\langle \kappa \rangle_\delta \frown \langle \kappa^{\mathrm{cf}(\delta)} \rangle \in \mathcal{C}(\delta + 1)$ (see Theorem 2.1).

Also, we shall prove that if κ is a singular cardinal of cofinality ω, then for every cardinal λ such that $\kappa < \lambda \leq \kappa^\omega$ we have that $\langle \kappa \rangle_\kappa \frown \langle \lambda \rangle_{\omega_2} \in \mathcal{C}(\kappa + \omega_2)$.

On the other hand, without using large cardinals, we shall obtain that it is relatively consistent with ZFC that for every singular cardinal κ of cofinality ω and every cardinal λ such that $\kappa < \lambda \leq \kappa^\omega$, $\langle \kappa \rangle_\kappa \frown \langle \lambda \rangle_{\omega_3} \in \mathcal{C}(\kappa + \omega_3)$.

2. A construction of LCS spaces with a large top

In this section, our aim is to prove the following result in ZFC.

Theorem 2.1. (a) *If κ is an infinite cardinal and $\delta < \kappa^+$ is a limit ordinal such that $\kappa^{<\mathrm{cf}(\delta)} = \kappa$, then $\langle \kappa \rangle_\delta \frown \langle \kappa^{\mathrm{cf}(\delta)} \rangle \in \mathcal{C}(\delta + 1)$.*

(b) *If κ is an infinite cardinal and $\delta < \kappa^+$ is a successor ordinal, then $\langle \kappa \rangle_\delta \frown \langle \kappa^\omega \rangle \in \mathcal{C}(\delta + 1)$.*

The following results are immediate consequences of Theorem 2.1.

Corollary 2.2. *If κ is an infinite cardinal of cofinality ω, then $\langle \kappa \rangle_\kappa \frown \langle \kappa^\omega \rangle \in \mathcal{C}(\kappa + 1)$.*