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A new fractal dimension for curves based on fractal structures

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ABSTRACT

In this paper, we introduce a new theoretical model to calculate the fractal dimension especially appropriate for curves. This is based on the novel concept of induced fractal structure on the image set of any curve. Some theoretical properties of this new definition of fractal dimension are provided as well as a result which allows to construct space-filling curves. We explore and analyze the behavior of this new fractal dimension compared to classical models for fractal dimension, namely, both the Hausdorff dimension and the box-counting dimension. This analytical study is illustrated through some examples of space-filling curves, including the classical Hilbert's curve. Finally, we contribute some results linking this fractal dimension approach with the self-similarity exponent for random processes.

1. Introduction

The word *fractal*, which derives from the Latin term *frangere* (that means "to break"), provided a novel concept in mathematics since Benoît Mandelbrot first introduced it in the early eighties [22]. Since then, both the study and the identification of fractal patterns have become more and more important due to the large number of applications to diverse scientific fields where fractals have been found, including computation, physics, economics and statistics among them (see [12,14,18,19]). Moreover, there has also been a particular interest in the application of fractals to social sciences (see for example [10] and its

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references). Nevertheless, some interesting non-standard objects had appeared previously as *mathematical monsters*, due to their novel and counter-intuitive analytical properties. Thus, they were frequently shown as counterexamples or exceptional objects provided by some remarkable mathematicians including Peano and Hilbert space-filling curves [21,24].

Fractals objects have been studied from different points of view, and the main tool that has been applied to study them is the fractal dimension, since it is their main invariant which shows some useful information about their complexity and irregularities. In particular, topology allows the study of this class of non-linear objects by means of *fractal structures*. They were first sketched in [6] and then formally defined and applied in [1] to characterize non-Archimedeanly quasi-metrizable spaces. This concept has allowed to formalize some topics on fractal theory from both theoretical and applied points of view. A fractal structure is just a countable collection of coverings of a given subset which provides better approximations to the whole space as we explore deeper stages, called *levels*. Thus, if we analyze the definition of the box-counting dimension, then we can observe that fractal structures provide a suitable context where new models of fractal dimension can be developed.

On the other hand, given any patch of the plane, a plane-filling curve is a continuous curve which meets every point in that patch. Thus, though the Peano plane-filling curve appeared in 1890, the later Hilbert's curve results also quite interesting, since it has no self-intersections nor touching points at any stage of its construction (that will be explain in detail later by means of fractal structures). In this way, a wide variety of space-filling curves were studied after that, though the example proposed by Hilbert still remains one of the most famous, since he provided one of the first graphical visualizations of a fractal in his original 2-page paper *Über die stetige Abbildung einer Linie auf ein Flächenstück* (1891) [21]. This curve was first sketched during a mathematical annual meeting in Bremen (Germany), where Hilbert and G. Cantor (1845–1918) were working on the foundation of the German mathematical society.

Our main purpose is to introduce a new theoretical model of fractal dimension for any fractal structure that becomes especially appropriate to analyze fractal patterns on curves. Additionally, we will explore some interesting connections between that fractal dimension approach and the self-similarity exponent of random processes.

The organization of this paper is as follows. In Section 2, we recall some preliminary definitions, notations and results including box-counting dimension, Hausdorff dimension, fractal structures and fractal dimension for a fractal structure. In Section 3, we provide a new theoretical model of fractal dimension especially appropriate to explore both the complexity and the fractal pattern of curves which is based on a novel concept of an induced fractal structure. This new procedure presents some advantages with respect to the classical models that may be applied for the same purpose, since it takes also into account the underlying structure of the curve. In addition to that, this is calculated on the image set of the curve, in contrast to the Hausdorff dimension and the box-counting dimension which are both calculated for its graph. We also show some theoretical properties of this fractal dimension for curves. In Subsection 3.1, we provide a theorem which allows us to generate space-filling curves among other applications. In Subsection 3.2, we explain in detail how to iteratively approach both the classical Hilbert's curve and a modified Hilbert's curve using fractal structures. We also calculate, compare and explain the values obtained for their classical fractal dimensions as well as for their new fractal dimensions. These examples show that the new model of fractal dimension we introduce in this paper results more accurate to distinguish and classify space-filling curves generated through different ways of construction. A curve which fills the whole Sierpiński's gasket is also explored from the point of view of fractal dimension (see Subsection 3.3). Finally, in Section 4, we show some theoretical results connecting the fractal dimension with the self-similarity exponent of random processes.