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Power bounded composition operators on spaces of meromorphic functions ☆

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ABSTRACT

We study composition operators with holomorphic symbols defined on spaces of meromorphic functions, when endowed with their natural locally convex topology. First, we show that such operators are well-defined, continuous and never compact. Then, we study the dynamics and prove that a composition operator is power bounded or mean ergodic if and only if the symbol is a nilpotent element in the group of automorphisms.

1. Introduction and notation

Let U be a open connected subset (=*domain*) of $\mathbb{C}$ and $\varphi : U \to U$ a holomorphic map of U into itself. The purpose of this brief note is to study the behavior of the orbits of composition operators $C_\varphi(f) := f \circ \varphi$, on the space $M(U)$ of meromorphic functions defined on U. We are interested in the case when the orbits of all the elements under C_φ are bounded. If this happens, the operator C_φ is called *power bounded*. We prove that, in this case, it is equivalent to C_φ be *mean ergodic*.

Given a subset $D \subset U$ we say that it is *discrete in* U whenever its accumulation set is contained in $\mathbb{C} \setminus U$ (i.e. it is discrete and closed in U). A *meromorphic function* f in U is a complex valued function f such

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that there exists a subset $D \subset U$ discrete in U with $f \in H(U \setminus D)$ and such that for each $u \in D$ there is $k \in \mathbb{N}$ such that $(z-u)^k f$ admits a holomorphic extension in u. The minimum value of k satisfying this condition is called *the order of f at u* and is denoted by $o_u(f)$. Let $P_f = D$ denote the set of *poles* of the meromorphic function f.

One natural way of endowing the space $M(U)$ of meromorphic functions with a topology is to consider it as a subspace of $C(U, \widehat{\mathbb{C}})$, $\widehat{\mathbb{C}}$ being the Alexandroff compactification of $\mathbb{C}$. In $\widehat{\mathbb{C}}$ it is considered the chordal metric and in $C(U, \widehat{\mathbb{C}})$ it is considered the topology τ_{chor} of locally uniform convergence. This is a metrizable non-locally convex topology, and moreover, a result of Cima and Schober [7] asserts that no comparable topology with τ_{chor} in $M(U)$ is complete. In 1995, Grosse-Erdmann studied deeply in [8] the locally convex topology introduced by Holdgrün in [9], giving a complete description of the seminorms, the properties of the topology, and showing that this topology, namely τ_{hol}, solved in the affirmative a conjecture of Tietz [15]. We describe briefly this topology.

A *positive divisor* δ on U is a map $\delta : U \to \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ such that $P_\delta = \{z \in U : \ \delta(u) \neq 0\}$ is discrete in U. As a consequence of the Laurent integral formula, the space

$$M(U, \delta) = \{f \in M(U) : \ P_f \subset P_\delta \text{ and } o_z(f) \leq \delta(z) \text{ for all } z \in P_\delta\}$$

is a closed subspace of $H(U \setminus P_\delta)$ endowed with the compact open topology. In fact, it is shown in [11] that $M(U, \delta)$ is isomorphic to $H(U)$. Hence it is Fréchet Montel (see Ref. [4]). We denote by $PD(U)$ the set of positive divisors on U, endowed with the natural order $\delta_1 \leq \delta_2$ when $\delta_1(u) \leq \delta_2(u)$ for all $u \in U$. The Holdgrün topology τ_{hol} is defined as the inductive limit

$$\operatorname*{ind}_{\delta \in PD(U)} M(U, \delta),$$

with respect to the inclusions $M(U, \delta_1) \hookrightarrow M(U, \delta_2)$ whenever $\delta_1 \leq \delta_2$. Grosse-Erdmann showed in [8] that endowed with this topology $M(U)$ is an ultrabornological, Montel and complete Hausdorff locally convex space. He also proved that each $M(U, \delta)$ is a topological subspace of $M(U)$, and hence, in particular, $H(U)$ endowed with the compact open topology (that we will denote by τ_c throughout this paper) is a closed topological subspace of $M(U)$, and that $M(U)$ is not separable. The product in $M(U)$ is separately but not jointly continuous, hence it is not an algebra. The projections over the terms in the Laurent development are continuous. Altogether these facts permit us to assert that Holdgrün's topology is the natural locally convex topology in $M(U)$. A fundamental system of seminorms was also given. However, we do not need to write them explicitly. In the following it will be important to note that a linear operator $T : M(U) \to M(U)$ is continuous if and only if the restriction of T to each step $M(U, \delta)$ is continuous. Moreover, each bounded set $B \subset M(U)$ is contained (and then bounded) in some step $M(U, \delta)$. The vector valued analogues of this topology have been studied in [6,10,11].

Let X be a locally convex Hausdorff space and $T : X \to X$ a continuous and linear operator from X to X. The iterates of T are denoted by $T^n := T \circ \cdots \circ T$, $n \in \mathbb{N}$. For $x \in X$ we write $Orb(T, x) := \{T^n x, n \in \mathbb{N}_0\}$ as the *orbit* of x by T. If the sequence $(T^n)_{n\in\mathbb{N}}$ is equicontinuous in the space $L(X)$ of all continuous and linear operators from X to X, T is called *power bounded*. In case $X = M(U)$, it is a Montel space, and hence barrelled. Consequently, an application of the uniform boundedness principle can be applied to conclude that T is power bounded if and only if the orbit $\{T^n(x) : \ n \in \mathbb{N}\}$ is bounded for every $x \in X$.

Given any $T \in L(X)$, we introduce the notation

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N},$$