Topology and its Applications 204 (2016) 1–7

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

# The Deligne groupoid of the Lawrence–Sullivan interval ☆

Urtzi Buijs, Yves Félix, Aniceto Murillo, Daniel Tanré

ARTICLE INFO

*Article history:*
Received 1 February 2016
Accepted 9 February 2016
Available online 27 February 2016

*Keywords:*
Lawrence–Sullivan inerval
Deligne groupoid
Maurer–Cartan elements

ABSTRACT

In this paper we completely describe the Deligne groupoid of the Lawrence–Sullivan interval as two parallel rational lines.



## 0. Introduction

Let MC($L$) be the set of Maurer–Cartan elements of a differential graded Lie algebra $L$ over $\mathbb{Q}$ which is assumed the base field henceforth. The group $L_0$ endowed with the Baker–Campbell–Hausdorff product, acts on MC($L$) as "a group of gauge transformations on flat connections" (see next section for precise and explicit terms). The groupoid associated to this action, known as the *Deligne groupoid*, was first introduced in [5] as a fundamental object to understand the Deligne principle by which every deformation functor is governed precisely by such a groupoid. See also [4,6].

On the other hand, *the Lawrence–Sullivan interval* [8] is a complete differential free graded Lie algebra $\mathfrak{L} = (\widehat{\mathbb{L}}(a, b, x), \partial)$ generated by two Maurer–Cartan elements $a$, $b$ and by an element $x$ of degree 0 joining $a$ and $b$ via the above action (see also next section). This particular object plays an essential role on the topological realization of (complete) differential graded Lie algebras [2,3] as well as on their homotopical behavior [1,10].

In this note, we explicitly describe the Deligne groupoid of $\mathfrak{L}$ and prove the following (see Theorem 2.3 for a precise statement):

**Theorem 0.1.** *The Deligne groupoid of $\mathfrak{L}$ is isomorphic to two disjoint copies of the rationals.*

☆ The first author has been partially supported by the Marie Curie COFUND programme U-mobility, co-financed by the University of Málaga, the European Commision FP7 under GA No. 246550, and MINECO (COFUND2013-40259).

The first and third authors have been partially supported by the Consejería de Economía, Innovación, Ciencia y Empleo, Junta de Andalucía grant FQM-213.

The fourth author has been partially supported by ANR-11-LABX-0007-01 "CEMPI".

The authors are partially supported by the MINECO grant MTM2013-41768-P.

*E-mail address:* ubuijs@uma.es (U. Buijs).

http://dx.doi.org/10.1016/j.topol.2016.02.004
0166-8641/

This readily implies that its nerve is a simplicial set homotopy equivalent to two disjoint points, which retrieves by a different argument the known geometrical realization of the Lawrence–Sullivan interval [2, Ex. 5.6], [3, §4].

As another consequence we also obtain is that any perturbation of $\mathfrak{L}$ produces an isomorphic DGL.

**Theorem 0.2.** *Let* $z \in \mathrm{MC}(\mathfrak{L})$*. Then,* $(\widehat{\mathbb{L}}(a,b,x), \partial_z)$ *in which* $\partial_z = \partial + \mathrm{ad}_z$*, is isomorphic to* $\mathfrak{L}$*.*

## 1. Preliminaries

Recall that a graded Lie algebra is a $\mathbb{Z}$-graded vector space $L = \oplus_{p\in\mathbb{Z}} L_p$ endowed with a bilinear Lie bracket satisfying antisymmetry $[x,y] = -(-1)^{|x||y|}[y,x]$ and Jacobi identity

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0\,.$$

Here $|x|$ denotes the degree of $x$. A *differential graded Lie algebra* or DGL is a graded Lie algebra $L$ together with a linear derivation $\partial$ of degree $-1$ such that $\partial^2 = 0$.

A Maurer–Cartan element of a given DGL is an element $z \in L_{-1}$ satisfying $\partial z + \frac{1}{2}[z,z] = 0$. We denote by $\mathrm{MC}(L)$ the set of Maurer–Cartan elements. These are preserved by DGL morphisms. Given $z \in \mathrm{MC}(L)$, the perturbed derivation $\partial_z = \partial + \mathrm{ad}_z$ is again a differential on $L$.

The *completion* $\widehat{L}$ of a graded Lie algebra $L$ is the projective limit

$$\widehat{L} = \varprojlim_n L/L^n$$

where $L^1 = L$ and for $n \geq 2$, $L^n = [L, L^{n-1}]$. A Lie algebra $L$ is called *complete* if $L$ is isomorphic to its completion. The completion of the free Lie algebra generated by the graded vector space $V$ is denoted by $\widehat{\mathbb{L}}(V)$.

Given $L$ a complete DGL, the *gauge action* of $L_0$ on $\mathrm{MC}(L)$ determines an equivalence relation among Maurer–Cartan elements defined as follows (see for instance [9, §4] or Proposition 1.2 below): given $x \in L_0$ and $z \in \mathrm{MC}(L)$,

$$x \,\mathcal{G}\, z = e^{\mathrm{ad}_x}(z) - \frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}(\partial x).$$

Here and from now on, the integer 1 inside an operator will denote the identity. Explicitly,

$$x \,\mathcal{G}\, z = \sum_{i\geq 0} \frac{\mathrm{ad}_x^i(z)}{i!} - \sum_{i\geq 0} \frac{\mathrm{ad}_x^i(\partial x)}{(i+1)!}.$$

The *Deligne groupoid* of $L$ has $\mathrm{MC}(L)$ as objects, and elements $x \in L_0$ as arrows from $x \,\mathcal{G}\, z$ to $z$. Geometrically [7,8], interpreting Maurer–Cartan elements as points in a space, one thinks of $x$ as a flow taking $x \,\mathcal{G}\, z$ to $z$ in unit time. In topological terms, the points $z$ and $x \,\mathcal{G}\, z$ are in the same path component.

**Definition 1.1.** ([8]) *The Lawrence–Sullivan interval* is the complete free DGL

$$\mathfrak{L} = (\widehat{\mathbb{L}}(a,b,x), \partial),$$

in which $a$ and $b$ are Maurer–Cartan elements, $x$ is of degree 0 and

$$\partial x = \mathrm{ad}_x b + \sum_{n=0}^{\infty} \frac{B_n}{n!}\,\mathrm{ad}_x^n(b-a) = \mathrm{ad}_x b + \frac{\mathrm{ad}_x}{e^{\mathrm{ad}_x} - 1}(b-a) \qquad (1)$$