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Completely ω-balanced topological groups

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We show that a topological group G admits a homeomorphic embedding as a subgroup into a product of strongly metrizable groups if and only if G is completely ω-balanced. Using this fact we obtain a characterization of strongly δ-complete topological groups, i.e., closed subgroups of a product of strongly metrizable groups.

1. Introduction

Having in mind the papers [3,4], the authors initiated the study of strongly δ-complete groups [5], i.e., the class of topological groups that can be embedded as closed subgroups into a product of strongly metrizable groups.

The notion of strong δ-completeness is between strong realcompactness and strong Dieudonné-completeness (see [8,6]) in the following sense: every strongly realcompact group is strongly δ-complete and each strongly δ-complete group is strongly Dieudonné-complete.

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We introduce the concept of completely ω-balanced group and prove that to be completely ω-balanced is closed under taking subgroups and products. In addition, we show the following:

$$\omega\text{-narrow} \Rightarrow \text{completely } \omega\text{-balanced} \Rightarrow \omega\text{-balanced}$$

We prove that a topological group G admits a homeomorphic embedding as a subgroup into a product of strongly metrizable groups if and only if G is completely ω-balanced (Theorem 4.1). Using this fact we give a characterization of strong δ-completeness: a topological group is strongly δ-complete if and only if it is completely ω-balanced and G_δ-closed in its Raĭkov completion (see Theorem 4.6).

2. Preliminaries

Throughout the paper all spaces and topological groups are assumed to be Tychonoff (completely regular and Hausdorff). A family $\mathcal{U}$ of subsets of a set X is *star-countable* if each $U \in \mathcal{U}$ intersects only countably many members of $\mathcal{U}$. If $\mathcal{U}$ can be decomposed into a countable union of star-countable covers of X, we call it a *σ-star-countable cover of* X. A family $\mathcal{V}$ of subsets of a set X is a *weak refinement* of a cover $\mathcal{U}$ if it contains a subfamily which is a cover of X and a refinement of $\mathcal{U}$. A space X is called *strongly metrizable* if it has a σ-star-countable base. Clearly, every base for a space X is an open weak refinement of any cover of X. The following result is presented in [9].

Lemma 2.1. *([9, Lemma 5]) A metrizable space X is strongly metrizable if and only if every open cover of X has an open weak refinement which is σ-star-countable.*

Given a topological group G with identity e, the symbol $\mathcal{N}(e)$ denotes the family of open neighborhoods of e. A subset V of a topological group G is called *ω-good* if there exists a countable family $\gamma \subset \mathcal{N}(e)$ such that for every $x \in V$, we can find $W \in \gamma$ with $xW \subseteq V$. The ω-good subsets were defined in the class of paratopological groups [7]. The following result permits us to prove Theorem 4.1.

Lemma 2.2. *([7, Lemma 3.10]) Every topological group G has a local base at the neutral element consisting of ω-good sets.*

Given a topological group G with identity e, we denote by $\mathcal{N}^*(e)$ the family of ω-good sets containing the identity. By Lemma 2.2, the family $\mathcal{N}^*(e)$ is a local base for G at e.

A family $\mathcal{U}$ of non-empty subsets of a topological group G is *dominated* by a family $\gamma \subseteq \mathcal{N}(e)$ if for every $U \in \mathcal{U}$ and $x \in U$ there exists $V \in \gamma$ such that $xV \subseteq U$.

Remember that a topological group G is called *ω-narrow* if for each $V \in \mathcal{N}(e)$ there exists a countable subset $A \subset G$ such that $AV = VA = G$. A topological group is *ω-balanced* if for each $U \in \mathcal{N}(e)$ there exists a countable subfamily γ of $\mathcal{N}(e)$ such that for each $g \in G$ there exists $V \in \gamma$ satisfying $gVg^{-1} \subset U$. The classes of ω-narrow and ω-balanced topological groups are closed under taking subgroups and products. The following results are known as Katz's theorem and Guran's theorem, respectively (see [1]).

Theorem 2.3. *Let G be a topological group. Then the following conditions are equivalent:*

a) *G is ω-balanced (respectively, ω-narrow);*
b) *G admits an embedding as a subgroup into a product of metrizable (respectively, separable metrizable) groups.*

All of other concepts and results about topological spaces and groups considered in this paper can be consulted in [2] and [1].